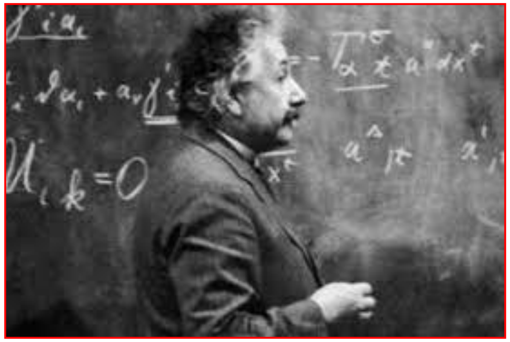
General Relativity (GR)

M1 - Physique 2023-2024



AE+GR (1907-1917)

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$$g_{\mu\nu}$$

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

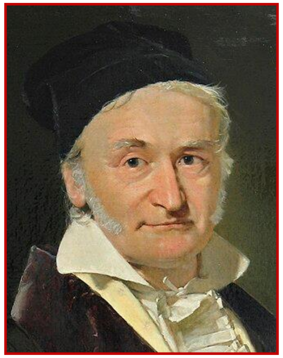
II) preliminaries of physics and mathematics

- …
- Special relativity, and complements
- Riemannian spaces & differential geometry

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

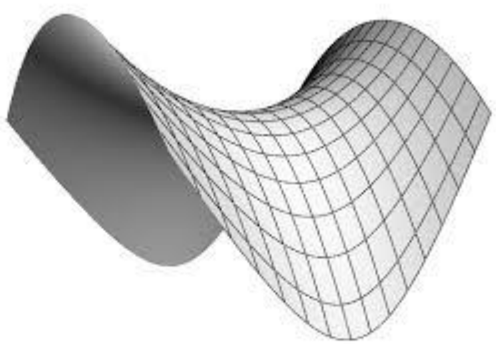
Curved surfaces

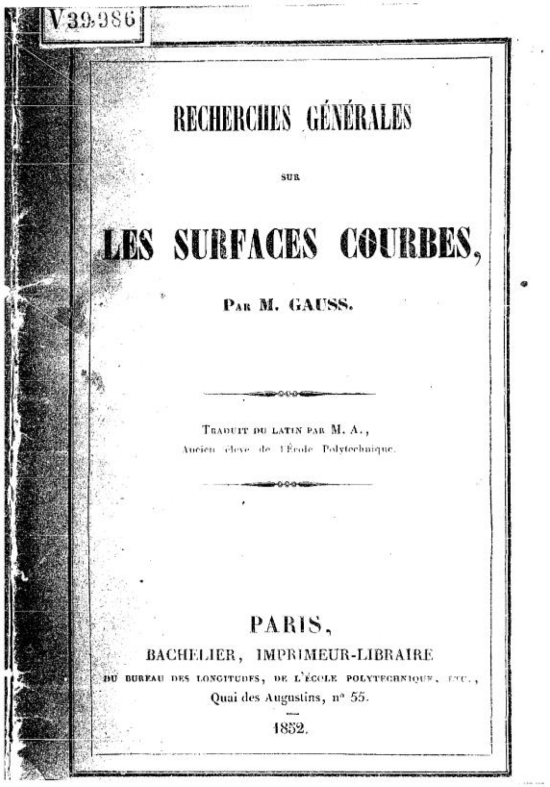


Carl Friedrich Gauss

(1777-1855)

Disquisitiones generales circa superficies curvas, 1828
Recherches générales sur les surfaces courbes
General research on curved surfaces





RECHERCHES GÉNÉRALES

SUR

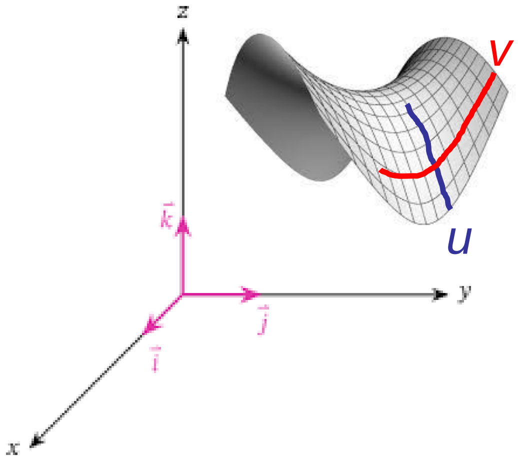
LES SURFACES COURBES,

PAR M. GAUSS.

TRADUIT DU LATIN PAR M. A.,
Ancien élève de l'École Polytechnique.

PARIS,
BACHELIER, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE, ETC.,
Quai des Augustins, n° 55.
1852.

Curved surfaces



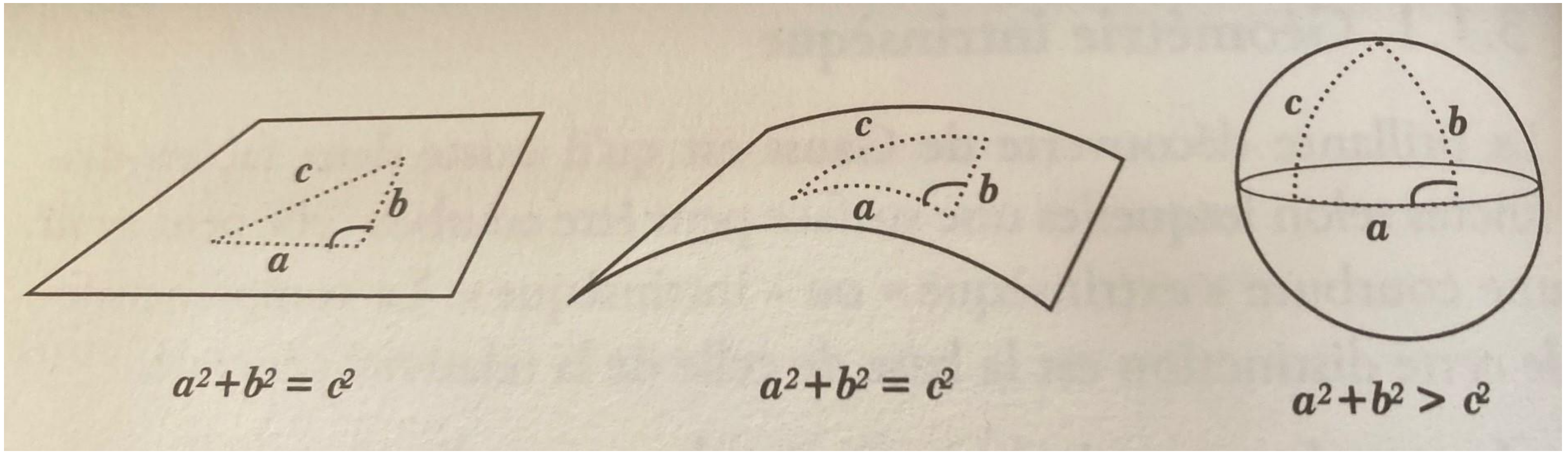
2d surface immersed in a 3d Euclidean space

In mathematics, differential surface geometry is the branch of differential geometry that deals with surfaces (the geometric objects of the usual E_3 space, or their generalization as 2-dimensional varieties), possibly equipped with additional structures, most often a Riemannian metric. In addition to the classical surfaces of Euclidean geometry (spheres, cones, cylinders, etc.), surfaces naturally appear as graphs of functions of two variables, or in parametric form, as sets described by a family of curves in space. Surfaces have been studied from various points of view: extrinsically, by focusing on their embedding in Euclidean space, and **intrinsically**, by focusing only on properties that can be determined from distances measured along curves drawn on the surface. One of the fundamental concepts discovered in this way was Gaussian curvature, studied in depth by Carl Friedrich Gauss (between 1825 and 1827), who demonstrated its **intrinsic** character.

Extrinsically and intrinsically curved

A triange on a plane surface

A triange on a sphere



A triange on a curved surface
folded sheet of paper

Pythagore: OK **OK** **not OK**

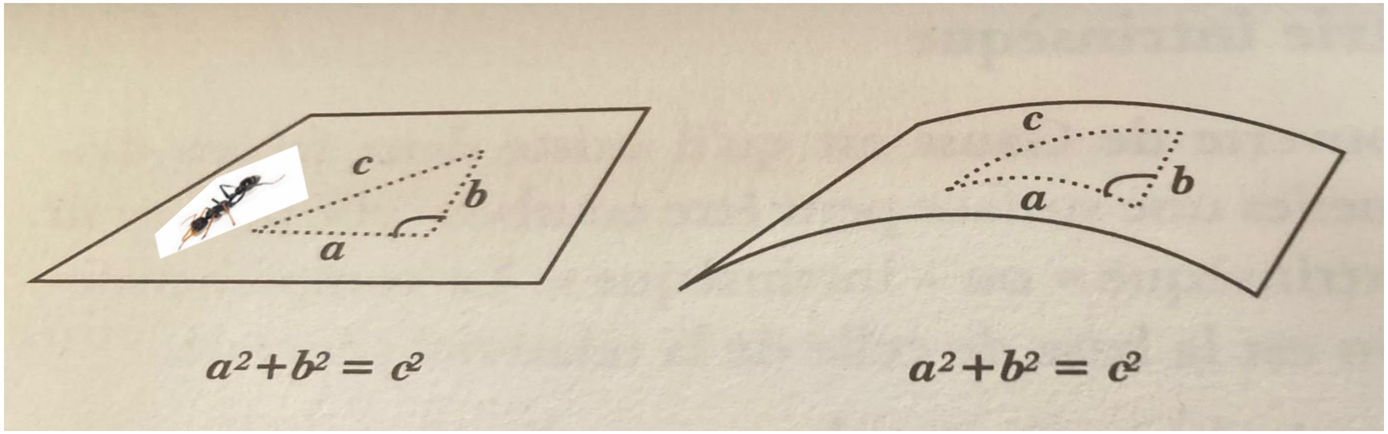
The surface of the central panel and the sphere are both extrinsically curved, but the former is intrinsically flat while the latter is intrinsically curved.

Extrinsically and intrinsically curved

A triange on a plane surface

A triange on a curved surface

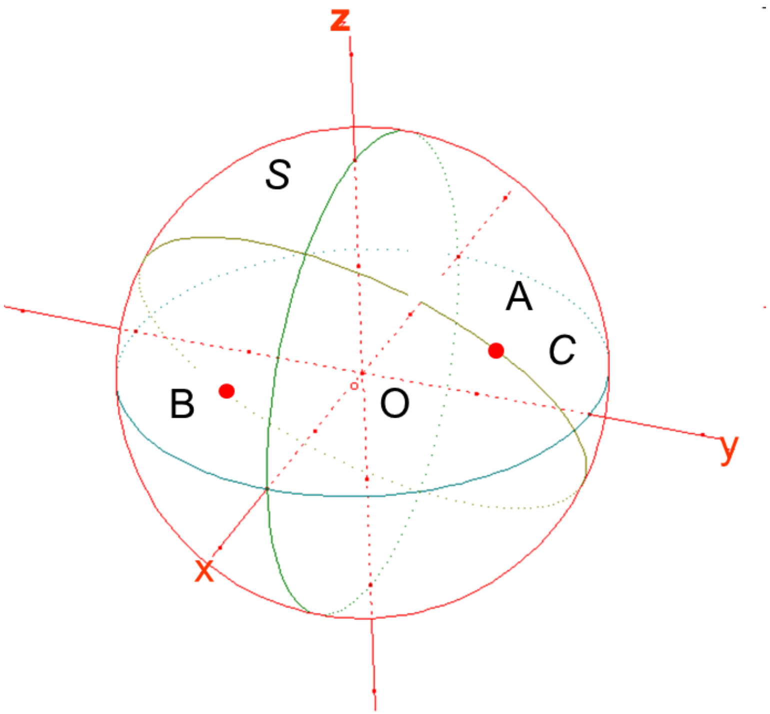
folded sheet of paper



If you were a tiny ant moving across the folded paper, able to measure angles and lengths of lines on the surface, but unable to look outside the paper, it would have no way of understanding that the surface on which it resides is not a plane. The two-dimensional geometry defined by the lengths of the intrinsic straight lines is the same as that of the plane. This geometry is called intrinsic geometry. This is why we say that the intrinsic geometry of the folded sheet of paper is flat, even though the paper itself is actually curved.

Extrinsically and intrinsically curved

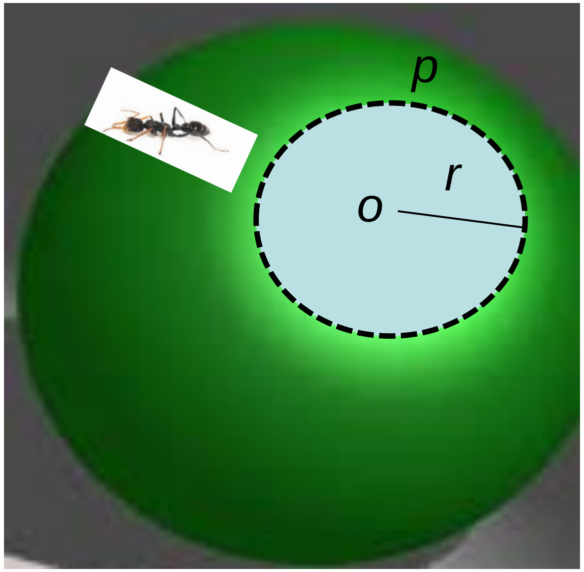
Given two points on the sphere, the shortest line connecting them on the surface is a portion of a great circle. These are the intrinsically straight segments of the sphere.



The shortest way to get from the North Pole to the South Pole is to follow a meridian. More generally, any great circle (i.e. the intersection of the sphere with a plane passing through center O) defines a geodesic of the sphere, and conversely any geodesic of the sphere is an arc of a great circle.

So straight lines on the sphere define an intrinsic geometry that differs from the geometry of the 2d plane.

Extrinsically and intrinsically curved



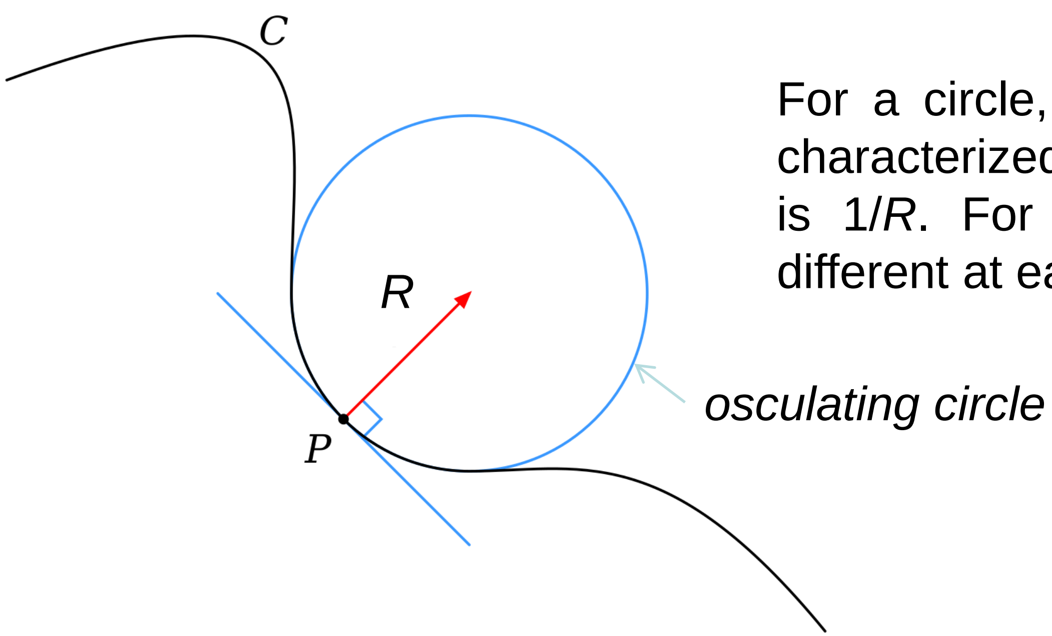
$$p \neq 2\pi r$$

An ant moving on the sphere and able to measure the lengths of lines on the surface but unable to look outside the surface will be able to understand that it is not on a plane. To do this, all you'd have to do is measure the length of the line drawn by all the points at distance *r* from a center *O*: if $p \neq 2\pi r$ the intrinsic geometry is not flat. When the geometry is not flat, the surface is said to have an intrinsic curvature.

Curvature, osculating circle

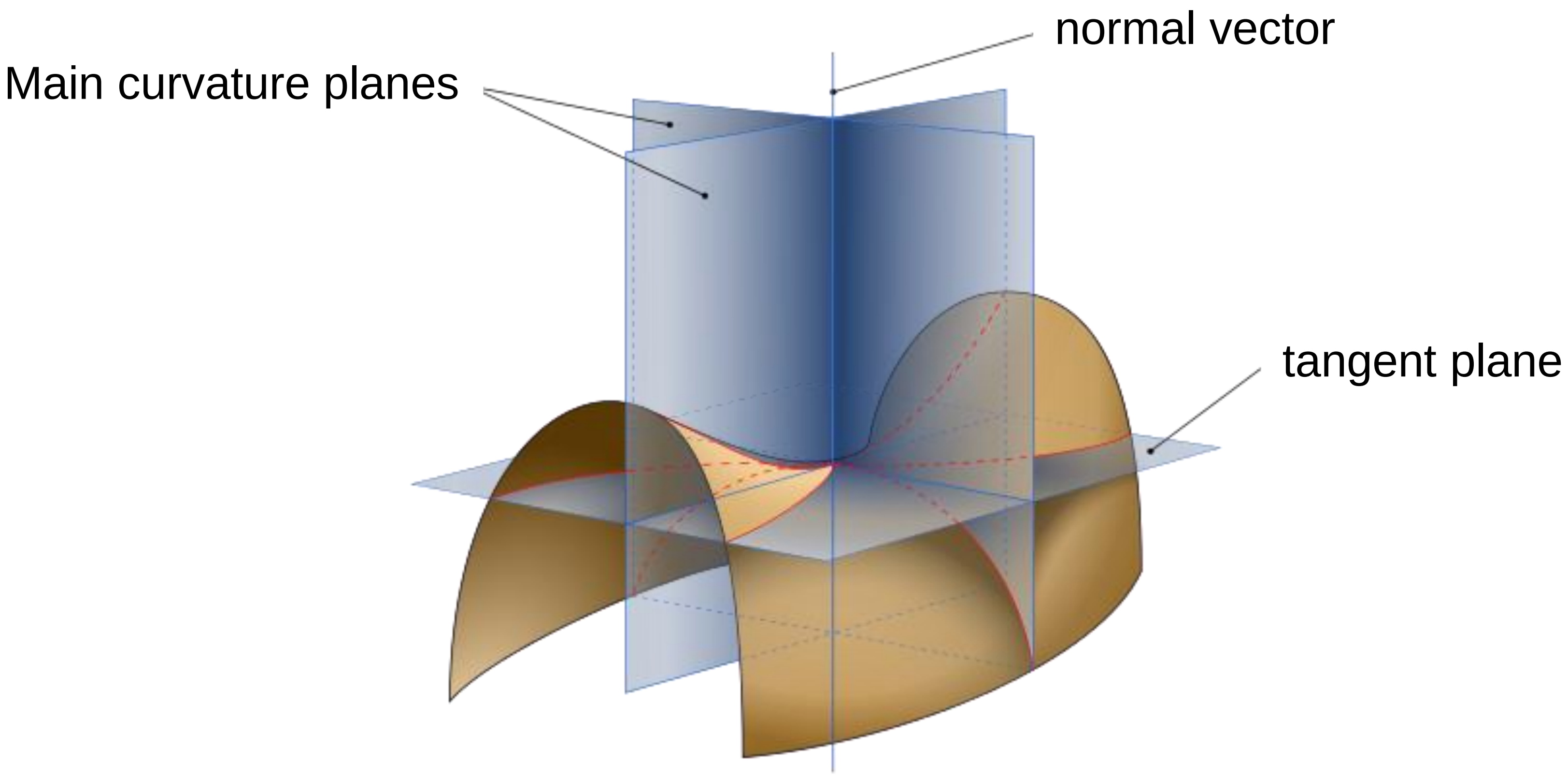
The osculating circle of a curve *C* at a given point P is **the circle that has the same tangent as at point as well as the same curvature *K***. Just as the tangent line is the line best approximating a curve at a point P (it involves the value of the first derivative of the function representing the curve *C* at point P), the osculating circle is the best circle that approximates the curve (it involves the value of the second derivative of the function representing the curve *C* at point P).

K (at P) = $1/R$



For a circle, curvature is constant and is characterized by the radius of the circle. It is $1/R$. For any curve, the curvature is different at each point of the curve.

Gaussian Curvature



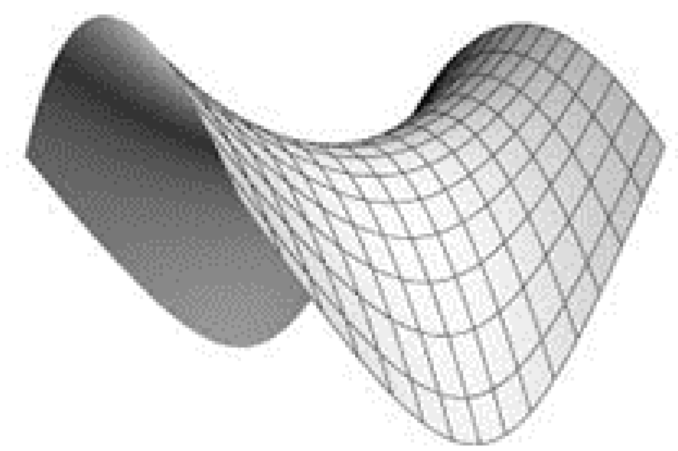
The minimum and maximum curvature values K_{min} and K_{max} are called principal curvatures. In general, they are different and, in this case, the planes corresponding to the two principal curvatures are perpendicular to each other. Their intersection with the tangent plane defines the principal directions. In the illustration opposite, the principal curvatures are of opposite sign, since one of the curves turns its concavity in the direction of the normal vector and the other in the opposite direction.

Gaussian Curvature

$K = K_{min} \times K_{max}$ *Gaussian curvature*

For a sphere of radius *R*, *K* is constant everywhere and is equal to:

$$K = 1/R \times 1/R = 1/R^2$$



Gaussian Curvature

Points on a surface are classified according to the **Gaussian curvature** of the surface at that point.

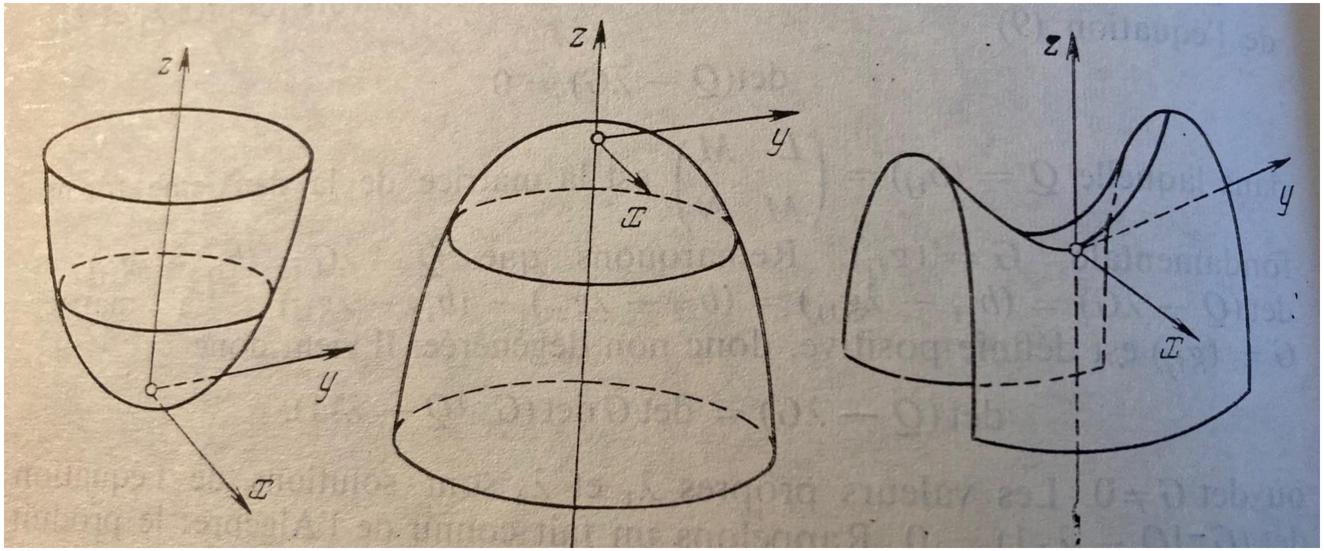
A point where the **Gauss curvature is strictly positive** is said to be **elliptical**. Such are the points of an ellipsoid, a two-sheet hyperboloid or an elliptical paraboloid. The two principal curvatures are of the same sign. If, moreover, they are equal, the point is an umbilicus. Such are the points of a sphere, or the two vertices of an ellipsoid of revolution.

A point **with zero Gaussian curvature** is said to be **parabolic**. At least one of the principal curvatures is zero. This is the case for points on a cylinder or cone, since the curvature along a generatrix of the cylinder passing through the point is zero*. This is also the case for any developable surface. If both principal curvatures are zero, the point is a flat. In the plane, all points are flats.

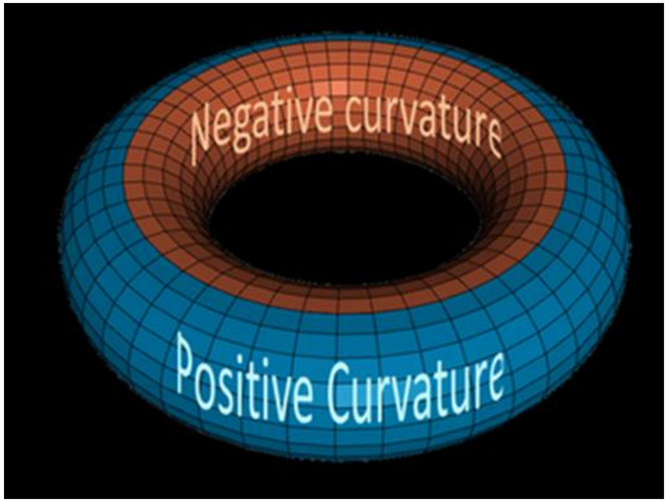
A point where the **Gaussian curvature is strictly negative** is said to be **hyperbolic**. At such a point, the two principal curvatures are of opposite sign. This is the case for all points of a one-sheet hyperboloid or hyperbolic paraboloid.

*A cylinder is intrinsically flat; example of a folded sheet of paper.

Gaussian Curvature



Elliptical ($K>0$) **Hyperbolic ($K<0$)**



Some points on the torus have positive curvature (elliptical points), while others have negative curvature (hyperbolic points).

Curved space

I. CURVILINEAR COORDINATES

Consider the transformation of one 4-dimensional coordinate system (we place ourselves directly in 4-dimensional space-time) into another one $(x^0, x^1, x^2, x^3) \to (x'^0, x'^1, x'^2, x'^3)$ we have

$$x^i = f^i(x'^0, x'^1, x'^2, x'^3) \ , \tag{1}$$

where f^i are certain functions.

Example (in 3d): $x^0 = x, x^1 = y, x^2 = z$ and $x'^0 = r, x'^1 = \theta, x'^2 = \varphi$. We have $(x = r\cos\theta\cos\varphi, x = r\cos\theta\sin\varphi, z = r\sin\theta)$.

The coordinate differentials are transformed according to the formulas

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k \ . \tag{2}$$

We call contravariant quadrivector any set of four A^i quantities which are transformed in a change of coordinates as the differentials of those coordinates. Thus,

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k \ . \tag{3}$$

Curved space

Similarly we have

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \quad (4)$$

$$A^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m} A'^{lm} \quad (5)$$

$$= \ldots \quad (6)$$

It is the natural generalization of quadrivectors and 4-tensors definitions in Galilean coordinates.

The square of the length element in curvilinear coordinates is a quadratic form of dx^i

$$ds^2 = g_{ij} dx^i dx^j \ , \quad (7)$$

where $g_{ij}(x^i)$ and $g_{ij} = g_{ji}$ ← *symmetric tensor*

Note: the contraction of g_{ij} by the contravariant tensor $dx^i dx^j$ gives a scalar. Therefore g_{ij} is a covariant tensor called the metric tensor. $ds'^2 = g'_{ij} dx'^i dx'^j = ds^2$.

Volume element

In Galilean coordinates we had: $d\Omega = dx^0 dx^1 dx^2 dx^3$. For curvilinear coordinates we have: $d\Omega' = \sqrt{-g} d\Omega$ with $g = \det[g_{ij}]$. It is related to the Jacobian of the transformation.

Covariant derivative

II. COVARIANT DERIVATIVE

In Galilean coordinates dA_i are the components of a vector (e.g $d\vec{r} = (dx, dy, dz)$) and $\frac{\partial A_i}{\partial x^k}$ are the components of a tensor.

In curvilinear coordinates it is not the same. This is because dA_i is the difference between vectors at different points in space, and vectors at different points in space transform differently since the coefficients in the transformation formulas are functions of the coordinates (e.g. $A^i = \frac{\partial x^i}{\partial x'^k} A'^k$).

proof

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \quad (8)$$

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\left(\frac{\partial x'^k}{\partial x^i}\right) \quad (9)$$

$$= \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l . \quad (10)$$

Therefore dA_i do not transform like a vector.

Covariant derivative

Note: If $\frac{\partial^2 x'^k}{\partial x^i \partial x^l} = 0$ then x'^k are linear functions of x^i (e.g. Lorentz transformation) and dA_i transform like a vector.

When comparing two vectors that are infinitely close, one must be transported parallel to the point when the other is located (in mathematics it is called parallel transport).

Let us consider a contravariant vector. A^i are its components at the point of coordinates x^i and $A^i + dA^i$ at the neighboring point; let's transport (parallel) the vector A^i to the point infinitely close $x^i + dx^i$; let δA^i be its increase. The difference DA^i between the two vectors located at the same position is

$$\boxed{DA^i \equiv dA^i - \delta A^i \; .} \tag{11}$$

Since the sum of two vectors must transform according to the same law, the increase must be linear in the components. So we have

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l \; , \tag{12}$$

where $\Gamma^i_{kl}(x^i)$:

Covariant derivative

- The form of Γ depend on the coordinate system.

Γ^i_{kl} is called the connection associated with the metric

- For a Galilean coordinate system $\Gamma = 0$.
- Γ^i_{kl} is not a tensor! A tensor that is zero in one coordinate system is zero in any other coordinate system So that we have $\Gamma'^i_{kl} \neq \frac{\partial x'^i}{\partial x^j}\frac{\partial x^n}{\partial x'^k}\frac{\partial x^m}{\partial x'^l}\Gamma^j_{nm}$
- Γ^i_{kl} are called Christoffel symbols of the second kind

Note 1: Instead of Γ^i_{kl} and $\Gamma_{i,kl}$, we sometimes use $\left\{ \begin{matrix} kl \\ i \end{matrix} \right\}$ and $\left[\begin{matrix} kl \\ i \end{matrix} \right]$.

Note 2: Bear in mind that Christoffel symbols are not tensors. This is why we must differentiate between the two types of symbols.

We have

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l \qquad (13)$$

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il}A_k\right)dx^l\ . \qquad (14)$$

proof

Let be A_i and B^i. We have $\delta(A_iB^i) = 0$. Therefore

$$B^i\delta A_i = -A_i\delta B^i = \Gamma^i_{kl}B^kA_idx^l \qquad (15)$$

$$B^i\delta A_i = \Gamma^k_{il}A_kB^idx^l\ , \qquad (16)$$

this is true whatever B^i leading to $\delta A_i = \Gamma^k_{il}A_kdx^l$. We define also

Christoffel symbols of the first kind

$$\Gamma_{i,kl} = g_{im}\Gamma^m_{kl}\ . \qquad (17)$$

Covariant derivative

In $DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k\right) dx^l$ the expression in brackets is a tensor since its product with dx^l (a vector) gives a vector.

Generalization of the notion of derivatives. These tensors are called covariant derivatives of the vectors A^i and A_i. We define

$$\boxed{DA^i \equiv A^i_{;l} dx^l \quad DA_i \equiv A_{i;l} dx^l} \,. \tag{18}$$

In Galilean coordinates $\Gamma^i_{kl} = 0$ leading to $DA^i = \frac{\partial A^i}{\partial x^l} dx^l$.

Covariant derivative of a tensor

$$DA^{ik} \equiv dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;l} dx^l \,, \tag{19}$$

with

$$A^{ik}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im} \,. \tag{20}$$

One also obtains

$$A^i_{k;l} = \frac{\partial A^i_k}{\partial x^l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k \tag{21}$$

$$A_{ik;l} = \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{im} \,, \tag{22}$$

Covariant derivative

> It is possible to generalize to all orders.

> If φ is a scalar field, since $\delta\varphi = 0 \Rightarrow D\varphi = d\varphi = \frac{\partial\varphi}{\partial x^k} dx^k = \partial_k \varphi dx^k$.

- $(A_i B_k)_{;l} = A_{i;l} B_k + A_i B_{k;l}$.
- $A_i^{;k} = g^{kl} A_{i;l}$ and $A^{i;k} = g^{kl} A^i_{;l}$.
- $\Gamma^i_{kl} = \Gamma^i_{lk}$ and $\Gamma_{i,kl} = \Gamma_{i,lk}$.
- There are 40 different quantities Γ^i_{kl}.
  - $n^2 - \frac{n(n-1)}{2}$ different quantities of a rank-2 symmetrical tensor
  - 4 times the previous result (for *n*=4; 4x10=40)

proof of $\Gamma^i_{kl} = \Gamma^i_{lk}$:

Since $A_{i;k}$ is a tensor it implies that $A_{k;i} - A_{i;k}$ is also a tensor. Suppose $A_i = \frac{\partial\varphi}{\partial x^i}$. We have $\frac{\partial A_i}{\partial x^k} = \frac{\partial^2\varphi}{\partial x^i \partial x^k} = \frac{\partial A_k}{\partial x^i}$. Using the definition of the covariant derivative we have

$$A_{k;i} - A_{i;k} = \left(\Gamma^l_{ik} - \Gamma^l_{ki}\right) \frac{\partial\varphi}{\partial x^l} \,. \qquad (23)$$

In Galilean coordinates the covariant derivatives are the ordinary derivatives i.e. the first member of the above equation is null. But since $A_{k;i} - A_{i;k}$ is a tensor its nullity in one coordinate system implies its nullity in any other system.

Covariant derivative

Christoffel symbols under a change of coordinates (without proof).

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m} \ . \qquad (24)$$

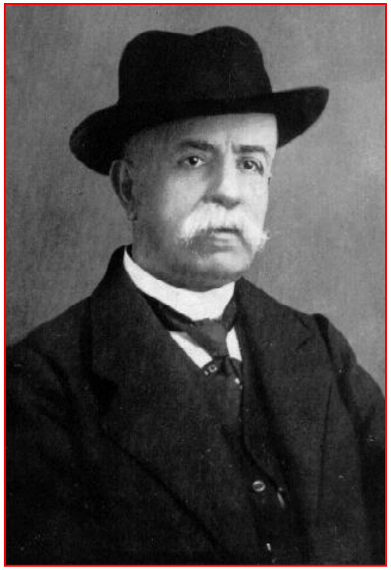
Link between the Christoffel symbol and the metric tensor.

We have

$$Dg_{ij} = 0 \ . \qquad (25)$$

It is the Ricci's theorem.

Gregorio Ricci



(1853-1925)

proof We have $DA_i = g_{ik} DA^k$ and $A_i = g_{ik} A^k$. It implies

$$DA_i = D\left(g_{ik} A^k\right) = g_{ik} DA^k + A^k Dg_{ik} \ . \qquad (26)$$

Comparing with $DA_i = g_{ik} DA^k$ it leads to

$$A^k Dg_{ik} = 0 \quad \forall A^k \qquad (27)$$

$$\Rightarrow Dg_{ik} = 0 \qquad (28)$$

$$\Rightarrow g_{ik;l} = 0 \ . \qquad (29)$$

Covariant derivative

We can therefore express Christoffel's symbols in terms of the derivation of the metric tensor. Let's write these derivatives by circularly permuting the indices i, k, l:

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl} \tag{30}$$

$$\frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik} \tag{31}$$

$$-\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li} \; . \tag{32}$$

(1829-1900)*

By using $\Gamma_{i,kl} = \Gamma_{i,lk}$ we obtain

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right) , \tag{33}$$

and with $\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}$ we have

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) . \tag{34}$$

* Professor at the University of Strasbourg from 1872 until his retirement in 1894.

Covariant derivative

Note 1: In cartesian coordinates $\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$ is an antisymmetric tensor (e.g. F_{ik} in *em*). In curvilinear coordinates this tensor reads $A_{i;k} - A_{k;i}$. However, given that $\Gamma^i_{kl} = \Gamma^i_{lk}$ and using the definition of the covariant derivative we have that

$$\boxed{A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\ .} \tag{35}$$

Note 2: It is easy to generalize the electromagnetic field equations of special relativity (SR) so that they can applied in any 4-dimensional curvilinear coordinate system, i.e. in the presence of a gravitational field. In SR $F_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$ leading in GR to $F_{ik} = A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$. Thus F_{ik} and A_i do not change. Therefore the first group of Maxwell's equations conserves its form: $F_{ik;l} + F_{li;k} + F_{kl;i} = 0 \Leftrightarrow \epsilon^{iklm}\frac{\partial F_{lm}}{\partial x^k} = 0$. For the second group, things are slightly more complex.

Lorentz equation

$$\text{(SR)}\ \ \frac{dp^i}{d\tau} = qF^{ik}u_k \ \ \Rightarrow \text{(GR)}\ \ \frac{Dp^i}{d\tau} = qF^{ik}u_k \tag{36}$$

$$\Rightarrow \text{(GR)}\ \ \frac{dp^i}{d\tau} + \Gamma^i_{kl}u^k u^l = qF^{ik}u_k \tag{37}$$

Geodesics

Motion of a particle in a gravitational field

In SR, the motion of a free material particle is determined by the principle of least action

$$\delta S = -mc\delta \int ds = 0 \; . \tag{38}$$

Recall that the distance between two events (1 and 2) in Minkowski space-time is given by $s_{12} = [c^2(t_2 - t_1) - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2]^{1/2}$ and $\Delta\tau = \tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\sqrt{1 - \beta^2} = \frac{1}{c}\int ds$. We have also $ds = cd\tau = cdt/\gamma$ where $d\tau$ is the infinitesimal proper time in the moving frame of reference.

By writing the Euler-Lagrange equation, we can find the equation of motion. Instead, it's simpler to find the equation of motion of a particle in the gravitational field by suitably generalizing the differential equations of free motion of a particle in SR i.e. in a 4-dimensional Minkowski space. These equations can be expressed as follows:

$$d\bar{u}^i/ds = 0 \Rightarrow d\bar{u}^i = 0 \; , \tag{39}$$

where $\bar{u}^i = dx^i/ds$ is the normalized 4-velocity ($\bar{u}^i = u^i/c$). In the present case (within the framework of SR), this corresponds to a straight line. In the ordinary 3d-space this translates into a uniform rectilinear motion (acceleration is zero i.e. $d\vec{v}/dt = 0$, so speed is constant).

Geodesics

Clearly, in curvilinear coordinates, this equation can be generalized as follows:

$$D\bar{u}^i = 0 \Rightarrow d\bar{u}^i + \Gamma^i_{kl}\bar{u}^k dx^l = 0 \qquad (40)$$

$$\Rightarrow \frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \qquad (41)$$

$$\Rightarrow \frac{d^2x^i}{d\tau^2} + \Gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0\ . \qquad (42)$$

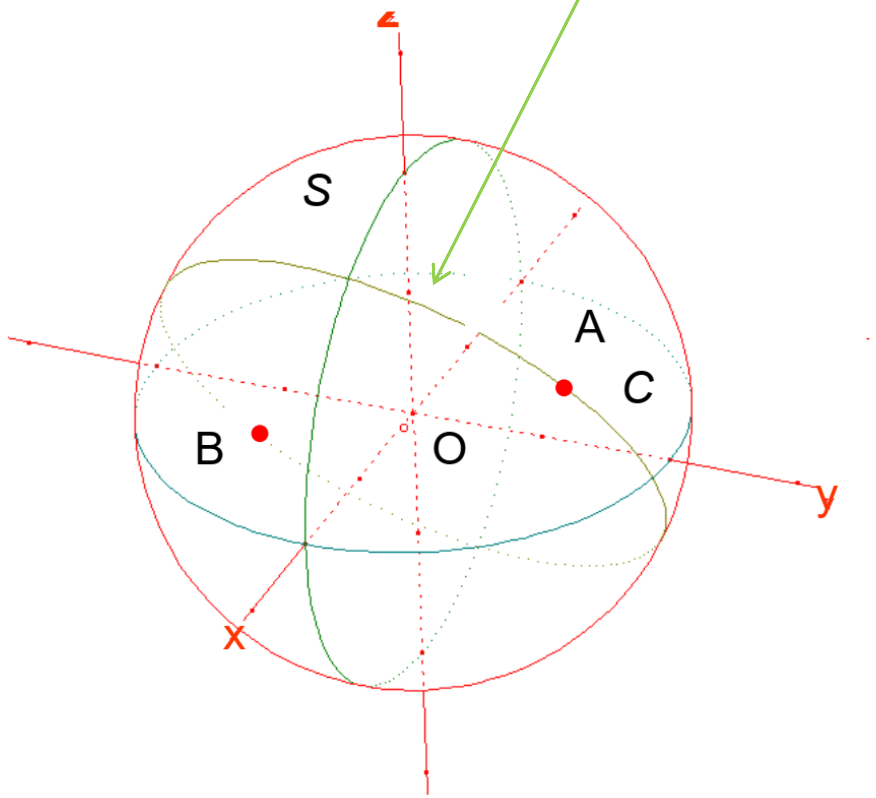
$\frac{d^2x^i}{ds^2}$ is the 4-acceleration of the particle. Therefore one can call the quantity $-m\Gamma^i_{kl}u^k u^l$ the 4-force acting on the particle in the gravitational field. The tensor g_{ij} plays the role of the potentials of the gravitation field: its derivatives define the Γ^i_{kl} field.

- $\frac{d^2x^i}{ds^2}$ is the 4-acceleration of the particle
- $-m\Gamma^i_{kl}u^k u^l$ the 4-force acting on the particle in the gravitational field
- The tensor g_{ij} plays the role of the potentials of the gravitation field

Note: The geodesic equation (42) can be obtained using the principle of least action, with *S* given by (38) and varying with respect to the metric tensor g_{ij}.

Geodesics

The geometric properties of figures drawn on the surface of a sphere are no longer those of Euclidean geometry. Thus, the shortest path from a point A to another point B, on the spherical surface, is constituted by an arc of a great circle passing through points A and B. For the sphere, great-circle arcs play the same role as straight lines in the plane. They are the sphere's geodesics.

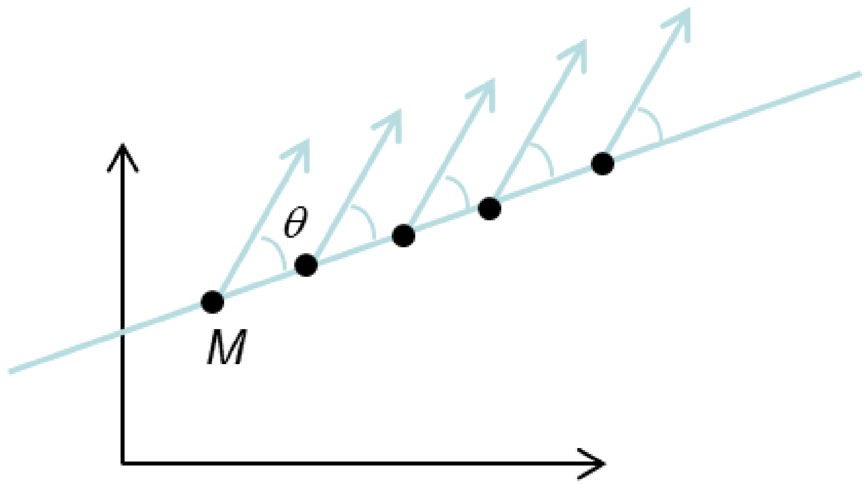


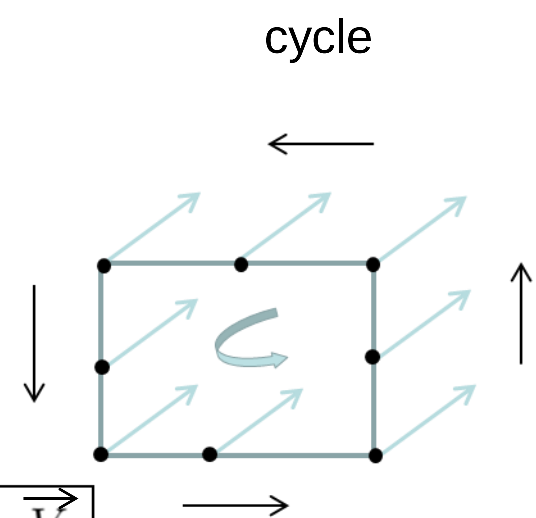
The shortest route from the North to the South Pole is to follow a meridian. More generally, any great circle (i.e. the intersection of the sphere with a plane passing through the center O) defines a geodesic of the sphere, and conversely any geodesic of the sphere is an arc of a great circle.

Parallel transport

Parallel transport, connections, curvature

Classical Euclidean geometry is based on the notions of parallelism and rigid displacements. A vector in a plane, defined at a point M on a line and making an angle θ with the line, retains the angle it makes with the line if moved parallel to itself along the line.



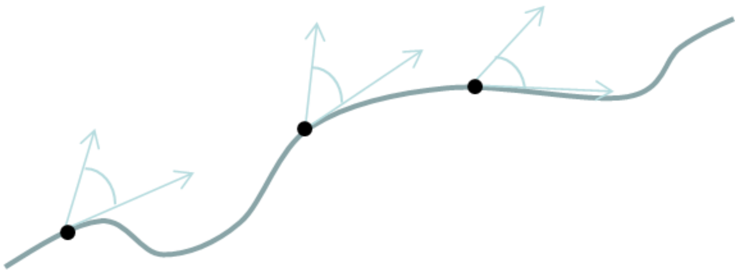


$\overrightarrow{V_1} = \overrightarrow{V_2}$

Parallel transport

If $x^i = x^i(s)$ are the parametric equations of a curve (s being the curvilinear abscissa measured from a given point) then the vector $v^i = dx^i/ds$ is the unit vector carried by the tangent to the curve. If the curve under consideration is a geodesic, we have $Dv^i = 0$ along this curve. This means that if we transport the vector v^i in parallel from a point x^i on the geodesic to another point $x^i + dx^i$ on the same geodesic, it coincides with the vector $u^i + du^i$ tangent to this line at the point $x^i + dx^i$. Consequently, in parallel transport along a geodesic, the tangent remains unchanged.

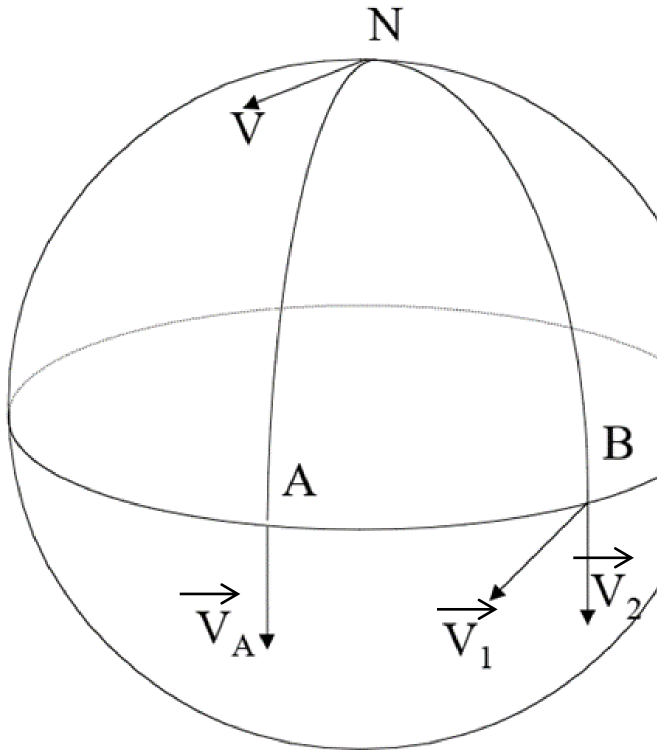
What's more, the angle between two vectors is clearly invariant when they are transported in parallel. We can therefore assert that during the transport of any vector along a geodesic curve, the angle between this vector and the tangent to this curve is constant.



geodesic

Parallel transport

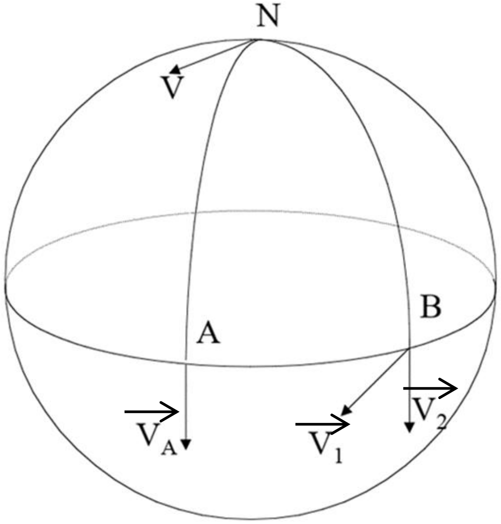
To illustrate parallel transport in any space, let's take the example of the sphere S^2. Consider a spherical triangle (whose sides are geodesics) formed by two half-meridians NA and NB and an equatorial arc AB, as shown in the figure below.



Parallel transport

We've seen that parallel transport of a vector along a geodesic preserves the angle it makes with it. If we move vector $\vec{V}$, tangent to arc NA at N, along arc NA and then along arc AB, we obtain the vector $\vec{V}_2$.

Parallel transport of $\vec{V}$ along arc NB gives vector $\vec{V}_1$, which makes an angle with $\vec{V}_2$ equal to angle $A\hat{N}B$. The angle between $\vec{V}_1$ and $\vec{V}_2$ is not zero, as it would be if the same thing had been done in a plane. This example illustrates the role of the curvature of the surface and that the result of a parallel transport depends on the path followed.



$$\overrightarrow{V_1} \neq \overrightarrow{V_2}$$

Curvature

Let's establish the general formula determining the variation of a vector during its parallel transport along an infinitesimal closed contour.

This variation ΔA_k may be expressed as $\oint \delta A_k$ where the integral is taken from the given contour. Using $\delta A_i = \Gamma^k_{il} A_k dx^l$ we have

$$\Delta A_k = \oint \Gamma^i_{kl} A_i dx^l \; . \tag{43}$$

Note: The vector A_i under the integral sign varies as it travels along the contour.

Applying Stokes' theorem to the curvilinear integral, noting that the area bounded by the contour under consideration is an infinitesimal quantity Δf^{lm} (infinitesimal surface element, $dx^l \wedge dx^m$), we obtain

$$\Delta A_k = \frac{1}{2}\left[\frac{\partial\left(\Gamma^i_{km} A_i\right)}{\partial x^l} - \frac{\partial\left(\Gamma^i_{kl} A_i\right)}{\partial x^m}\right]\Delta f^{lm} \tag{44}$$

$$= \frac{1}{2}\left[\frac{\partial \Gamma^i_{km}}{\partial x^l} A_i - \frac{\partial \Gamma^i_{kl}}{\partial x^m} A_i + \Gamma^i_{km}\frac{\partial A_i}{\partial x^l} - \Gamma^i_{kl}\frac{\partial A_i}{\partial x^m}\right]\Delta f^{lm} \; . \tag{45}$$

Using $\frac{\partial A_i}{\partial x^l} = \Gamma^n_{il} A_n$ (related to $\delta A_i = \Gamma^k_{il} A_k dx^l$ must be admitted) we get

$$\Delta A_k = \frac{1}{2} R^i_{klm} A_i \Delta f^{lm} \; , \tag{46}$$

Curvature

where R^i_{klm} is a 4th-order tensor defined by

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \ . \tag{47}$$

The tensor character of $\mathbb{R}$ results from the fact that in the above expression of ΔA_k in the left-hand side we have a vector which is the difference of the values of a vector at one and the same point. The tensor is called the curvature tensor or Riemann tensor.

We can easily obtain an analogous formula for the contravariant vector A^k (or the contravariant components of a vector):

$$\Delta A^k = -\frac{1}{2} R^k_{ilm} A^i \Delta f^{lm} \ . \tag{48}$$

When we take the covariant derivative of a vector A_i with respect to x^k and x^l twice, the result of the derivation generally depends on the order of derivation, unlike for ordinary derivatives. In this case, we have

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{ikl} \ . \tag{49}$$

(*) $A_{i;k;l} \equiv A_{i;kl}$ and $A^i_{;k;l} \equiv A^i_{;kl}$ are the second covariant derivative. We have $A^i_{;k;l} = \nabla_l \left(\partial_k A^i + \Gamma^i_{jk} A^j\right)$ and $D^2 A^i = A^i_{;k;l} dx^k dx^l$.

Properties of the curvature tensor

- From (47) one has $R^i_{klm} = -R^i_{kml}$.
- We have: $R^i_{klm} + R^i_{mkl} + R^i_{lmk} = 0$.
- $R_{iklm} = g_{in} R^n_{klm}$ which leads to $R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right)$.
- $R_{iklm} = -R_{kilm} = -R_{ikml}$ and $R_{iklm} = R_{lmik}$. Therefore all components of R_{iklm} for which $i = k$ or $l = m$ are zero.
- $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.
- Bianchi identity: $R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0$.

Luigi Bianchi

(1856-1928)

Ricci tensor and scalar curvature

The Ricci tensor is defined as

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk} \; . \qquad (50)$$

Using (47) one get's

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} \; . \qquad (51)$$

This tensor is obviously symmetrical i.e. $R_{ik} = R_{ki}$.

Finally, by contracting R_{ik} one obtains the invariant

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm} \; . \qquad (52)$$

R is called the scalar curvature of space.

Riemannian space

Examples of Riemann spaces

Sphere - Consider a sphere of radius R, area S, located in ordinary three-dimensional space. The Cartesian coordinates x, y, z of a point M on the surface S can be expressed, for example, in terms of the spherical coordinates, longitude φ and colatitude θ. The sphere is fully described if $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

Two such parameters, used to determine a point on the surface of the sphere, are called curvilinear coordinates on the surface or Gaussian coordinates. Any other parameters, u, v, can of course be chosen as curvilinear coordinates on the surface.

The linear element of the surface ds^2, square of the distance between two infinitely neighboring points M and M', is written in terms of spherical coordinates, for R =constant :

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2 \ . \qquad (1)$$

We obtain an expression for the linear element as a function of just the two Gaussian coordinates θ and φ.

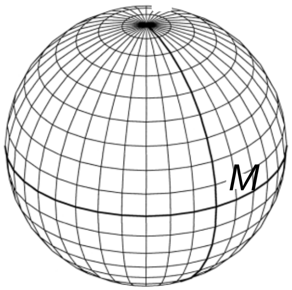
Riemannian fundamental tensor - Being described using two parameters, the surface of a sphere (considered as a two-dimensional space) is an example of a two-dimensional Riemann space

Riemannian space

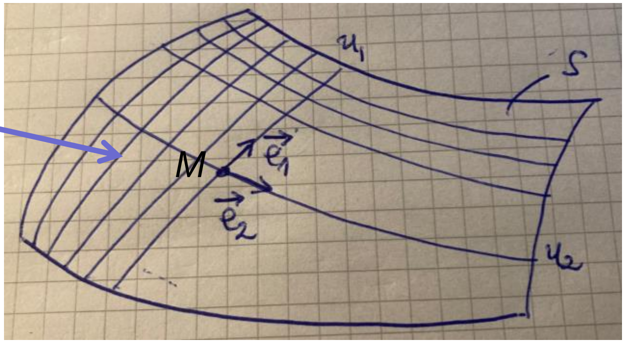
The linear element (1) is of the general form:

$$ds^2 = g_{ij} du^i du^j \ , \qquad (2)$$

where du^i are the contravariant components of the vector $d\vec{M} = M\vec{M}'$ with respect to the natural reference frame $(M, \vec{e}_i)$. Posing $u^1 = \theta$, $u^2 = \varphi$, we obtain by identification of formulas (1) and (2):



$$\begin{cases} g_{11} = R^2 \\ g_{12} = g_{21} = 0 \\ g_{22} = R^2 \sin^2\theta \ . \end{cases}$$



The quantities g_{ij} with $i, j = 1, 2$, constitute the components of a tensor which is the fundamental tensor of the Riemannian space formed by the surface S. This is an example of a Riemannian fundamental tensor or Riemannian metric.

Exercise 1: First of all replace R by a in the above expressions. From the g_{ij} values compute the six (justify why six) possible different $\Gamma_{i,kl}$ (there will be 40 in four dimensions). Then compute $\Gamma_{i,kl}$ and R^i_{klm}. How many elements of this tensor are non-zero? Finally compute the components of the Ricci tensor and the scalar curvature. Conclusions.

Riemannian space

Definition of a Riemann space

A Riemann space is a variety to which a metric has been attached. This means that, in each part of the variety, represented analytically by means of a coordinate system (u^i), we have given ourselves a metric defined by the quadratic form

$$ds^2 = g_{ij} du^i du^j \; . \tag{3}$$

The g_{ij} coefficients are not entirely arbitrary and must satisfy the following conditions:

- The g_{ij} components are symmetrical: $g_{ij} = g_{ji}$.
- The determinant of the matrix $[g_{ij}]$ is non-zero
- The differential form (3), and therefore the concept of distance defined by g_{ij}, is invariant to any change of coordinate system.
- All partial derivatives of order two of g_{ij} exist and are continuous (we say that g_{ij} are of class $C2$).

A Riemannian space is therefore a space of points, each marked by n coordinates u^i, endowed with any metric of the form (3) verifying the above conditions. This metric is called Riemannian.

If the metric is positive definite, i.e. $g_{ij} v^i v^j$, for any non-zero vector $\vec{v}$, the space is said to be properly Riemannian. In this case, the determinant of the matrix $[g_{ij}]$ is strictly positive and all its eigenvalues are strictly positive.

Riemannian space

How do we distinguish between a Euclidean and a Riemannian metric? First, let's define what we mean by a Euclidean metric.

We know that every Euclidean space has orthonormal basis such that $g_{ij} = \delta_{ij}$. By definition, a metric of a space is said to be Euclidean when any fundamental tensor of this space can be reduced, by an appropriate change of coordinates, to a form such that $g_{ij} = \delta_{ij}$. Thus, the fundamental tensors defined bv the linear elements (3) cannot be reduced to a Euclidean tensor. The definition of Riemannian spaces shows that Euclidean space is a very special case of such spaces. There is only one Euclidean space, whereas an infinite number of Riemannian spaces can be invented.

Let's ask the key question: when is intrinsic geometry defined by a flat $g_{ij}(x^i)$ metric field?

Certainly, the constant metric field $g_{ij}(x^i) = \delta_{ij}$ defines a flat metric, because in this case the distances are given by $ds^2 = \delta_{ij} dx^i dx^j$ which means that the coordinates x^i are the Cartesian coordinates of a Euclidean space. But if we introduce new coordinates x'^i the resulting metric $g'_{ij}(x'^i) = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \delta_{kl}$ is also flat, since the intrinsic geometry doesn't change by changing the coordinates. Hence the question: how do we know whether a given $g_{ij}(x'^i)$ defines a flat intrinsic geometry?

Riemannian space

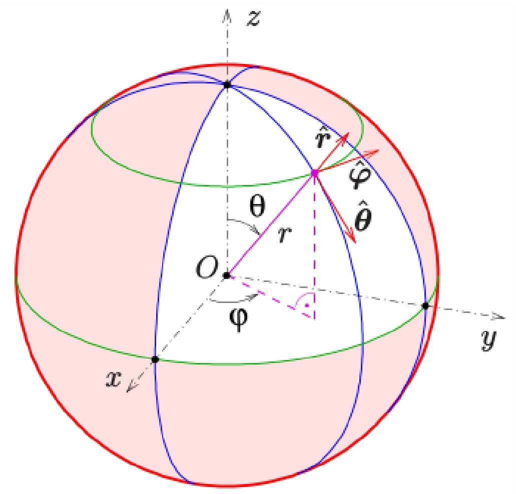
Now, if space is flat, we can choose global Cartesian coordinates. In these coordinates $g_{ij} = \delta_{ij}$ and the expression cancels out insofar as the quantities Γ cancel out, since they consist solely of derivatives of the metric, hence $R^m_{ikl} = 0$. Now R^m_{ikl} is a tensor $\mathbb{R}$: if it cancels out in one coordinate system, it cancels out in all the others. We have therefore found a way of testing whether the space is flat: it must satisfy $R^m_{ikl} = 0$.

Riemann was able to show that $R^m_{ikl} = 0$ is not only necessary but also sufficient for space to be flat. In other words, it is possible to put the metric into $g_{ij}(x^i) = \delta_{ij}$-form in a region. In more mathematical terms, $R^m_{ikl} = 0$ are the integrability conditions for solving $x^i = f^i(x'^i)$ when $g'_{ij}(x'^i)$ is given.

We'll call R^m_{ikl} the Riemann tensor or Riemann curvature tensor. It generalizes Gaussian curvature to the intrinsic geometry of spaces of arbitrary dimension. This is the beautiful result of Riemann's thesis.

Exercise 2: Same as Exercise 1 but for a three dimensional sphere (r, θ, φ). Conclusion.

Exercices



The line element for an infinitesimal displacement from r, θ, φ to $r + dr, \theta + d\theta, \varphi + d\varphi$ is

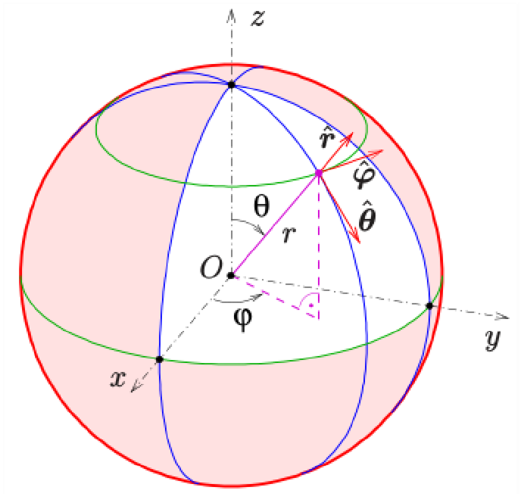
$$\mathrm{d}\vec{M} = \mathrm{d}r\,\hat{e_r} + r\,\mathrm{d}\theta\,\hat{e_\theta} + r\sin\theta\,\mathrm{d}\varphi\,\hat{e_\varphi}\ ,$$

where

$$\begin{cases} \hat{e_r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}, \\ \hat{e_\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}, \\ \hat{e_\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} \end{cases}$$

are the local orthogonal unit vectors in the directions of increasing r, θ, and φ, respectively, and $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors in Cartesian coordinates.

Exercices



> Note: $(\hat{e_r}, \hat{e_\theta}, \hat{e_\varphi})$ is not a local basis since $d\vec{M} \neq d\theta\hat{e_\theta} + d\varphi\hat{e_\varphi} + dr\hat{e_r}$. Therefore one must define new vectors: $\vec{e_1} = r\hat{e_\theta}, \vec{e_2} = r\sin\theta\hat{e_\varphi}, \vec{e_3} = \hat{e_r}$. In this case we have $d\vec{M} = du^i\vec{e_i}$ with $du^1 = d\theta, du^2 = d\varphi, du^3 = dr$.

> Using $d\vec{M} = du^i\vec{e_i}$ we have $\|d\vec{M}\|^2 = ds^2 = g_{ij}du^i du^j$ with $g_{ij} = \vec{e_i}\cdot\vec{e_j}$.

> Only the two-dimensional case doesn't produce monstrous calculations. The indices can only take two values, and the special cases derived from symmetry relations cover all components of the Riemann-Christoffel tensor.

> Suppose *r* = cte = *a*. We have *dr* = 0. Space of dimension 2.

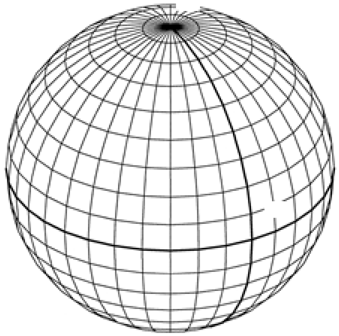
Exercices

$$u^1 = \theta \qquad u^2 = \varphi$$

Curvilinear coordinates

$$g_{11} = a^2 \qquad g_{22} = a^2 \sin^2\theta \qquad g_{21} = g_{12} = 0$$
$$g^{11} = 1/a^2 \qquad g^{22} = \frac{1}{a^2 \sin^2\theta} \qquad g^{21} = g^{12} = 0$$

Metric tensor



$$\begin{cases} \Gamma^1{}_{11} = 0 \quad \Gamma^2{}_{11} = 0 \\ \Gamma^1{}_{12} = 0 \quad \Gamma^2{}_{12} = \mathrm{cotan}\,\theta \\ \Gamma^1{}_{21} = 0 \quad \Gamma^2{}_{21} = \mathrm{cotan}\,\theta \\ \Gamma^1{}_{22} = -\sin\theta\,\cos\theta \quad \Gamma^2{}_{22} = 0 \end{cases}$$

Christoffel symbols

Using symmetry arguments we have:

$$\begin{cases} R^1{}_{111} = R^2{}_{111} = R^1{}_{222} = R^2{}_{222} = 0 \\ R^1{}_{211} = R^2{}_{211} = R^1{}_{122} = R^2{}_{122} = 0 \\ R^1{}_{112} = -R^1{}_{121} \\ R^2{}_{112} = -R^2{}_{121} \\ R^1{}_{221} = -R^1{}_{212} \\ R^2{}_{221} = -R^2{}_{212} \end{cases}$$

→ Four independent coefficients to calculate

Exercices

$$R^i{}_{ljk} = \partial_k \Gamma^i{}_{lj} - \partial_j \Gamma^i{}_{lk} + \Gamma^i{}_{mk}\,\Gamma^m{}_{lj} - \Gamma^i{}_{mj}\,\Gamma^m{}_{lk}$$

Expanding these summations one gets:

$$R^1{}_{112} = -R^1{}_{121} = \cdots$$
$$\partial_2 \Gamma^1{}_{11} - \partial_1 \Gamma^1{}_{12} + (\Gamma^1{}_{12}\,\Gamma^1{}_{11} + \Gamma^1{}_{22}\,\Gamma^2{}_{11}) - (\Gamma^1{}_{11}\,\Gamma^1{}_{12} + \Gamma^1{}_{21}\,\Gamma^2{}_{12})$$
$$\cdots = 0 - 0 + (0+0) - (0+0) = 0$$

$$R^2{}_{112} = -R^2{}_{121} = \cdots$$
$$\partial_2 \Gamma^2{}_{11} - \partial_1 \Gamma^2{}_{12} + (\Gamma^2{}_{12}\,\Gamma^1{}_{11} + \Gamma^2{}_{22}\,\Gamma^2{}_{11}) - (\Gamma^2{}_{11}\,\Gamma^1{}_{12} + \Gamma^2{}_{21}\,\Gamma^2{}_{12})$$
$$\cdots = 0 - \partial_\theta \operatorname{cotan}\theta + (0+0) - (0 + \operatorname{cotan}\theta\ \operatorname{cotan}\theta) = \frac{1-\cos^2\theta}{\sin^2\theta} = 1$$

$$R^1{}_{221} = -R^1{}_{212} = \cdots$$
$$= \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^1{}_{21} + (\Gamma^1{}_{11}\,\Gamma^1{}_{22} + \Gamma^1{}_{21}\,\Gamma^2{}_{22}) - (\Gamma^1{}_{12}\,\Gamma^1{}_{21} + \Gamma^1{}_{22}\,\Gamma^2{}_{21})$$
$$\cdots = \partial\theta\,(-\sin\theta\ \cos\theta) - 0 + (0+0) - [0 + (-\sin\theta\ \cos\theta)\ \operatorname{cotan}\theta]$$
$$\cdots = (-\cos^2\theta + \sin^2\theta) + \cos^2\theta = \sin^2\theta$$

$$R^2{}_{221} = -R^2{}_{212} = \cdots$$
$$\partial_1 \Gamma^2{}_{22} - \partial_2 \Gamma^2{}_{21} + (\Gamma^2{}_{11}\,\Gamma^1{}_{22} + \Gamma^2{}_{21}\,\Gamma^2{}_{22}) - (\Gamma^2{}_{12}\,\Gamma^1{}_{21} + \Gamma^2{}_{22}\,\Gamma^2{}_{21})$$
$$\cdots = 0 - 0 + (0+0) - (0+0) = 0$$

Exercices

Ricci tensor

$$\boxed{R_{jk} = R^m_{jlm}}$$

$$R_{11} = R^1{}_{111} + R^2{}_{112} = 0 + 1 = 1$$

$$R_{12} = R^1{}_{121} + R^2{}_{122} = 0 + 0 = 0$$

$$R_{21} = R^1{}_{211} + R^2{}_{212} = 0 + 0 = 0$$

$$R_{22} = R^1{}_{221} + R^2{}_{222} = \sin^2\theta + 0 = \sin^2\theta$$

Scalar curvature

$$\boxed{R = g^{mn} R_{mn}}$$

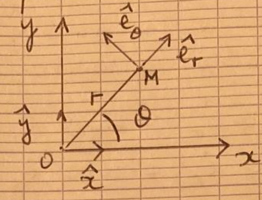
$$R = g^{11} R_{11} + g^{22} R_{22} = \frac{1}{a^2} \cdot 1 + \frac{1}{a^2 \sin^2\theta} \sin^2\theta = \frac{2}{a^2} \neq 0$$

→ **Riemannian space**

Exercices

Curvilinear coordinates

* $n = 2 \rightarrow$ polar coordinates



> $\|\hat{e}_r\| = \|\hat{e}_\theta\| = 1$
> $\hat{e}_r \cdot \hat{e}_\theta = 0$

$$\begin{cases} x^1 = x \\ x^2 = y \end{cases} ; \quad \begin{matrix} x'^1 = r \\ x'^2 = \theta \end{matrix} \qquad \begin{cases} x^1 = x'^1 \cos(x'^2) \\ x^2 = x'^1 \sin(x'^2) \end{cases}$$

$$x^i = f^i(x'^j).$$

$$J = \frac{\partial(x^1, x^2)}{\partial(x'^1, x'^2)} \quad \text{Jacobian}$$

$$\Rightarrow \text{Jacobian matrix} \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix} = \mathbb{J}$$

$$\mathbb{J} : \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} ; \quad J = \det \mathbb{J} = r$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$dy = d\theta\, r\cos\theta + \sin\theta\, dr$$

$$\Rightarrow dx \wedge dy = \underbrace{(r)}_{= J}\, dr \wedge d\theta$$

(Wedge product)

$(d\Omega = J\, d\Omega')$.
↳ infinitesimal volume element.

$d\vec{M}$

$\vec{M} \,\Big|\, \begin{matrix} x \\ y \end{matrix}$

$$\left.\begin{matrix} x \rightarrow x + dx \\ y \rightarrow y + dy \\ \vec{M} \rightarrow \vec{M} + d\vec{M} \end{matrix}\right\} \Rightarrow d\vec{M} = d(x\hat{x} + y\hat{y}) = dx\,\hat{x} + dy\,\hat{y}$$

since $d\hat{x} = d\hat{y} = 0$.

Exercices

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} dx = dr\cos\theta - r\sin\theta\, d\theta \\ dy = dr\sin\theta + r\cos\theta\, d\theta . \end{cases}$$

$$\begin{cases} \hat{e}_r = \cos\theta\, \hat{x} + \sin\theta\, \hat{y} \\ \hat{e}_\theta = -\sin\theta\, \hat{x} + \cos\theta\, \hat{y} \end{cases}$$

One checks $\hat{e}_r \cdot \hat{e}_\theta = 0$.

$$\Rightarrow \begin{cases} \hat{x} = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta \\ \hat{y} = \sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta \end{cases}$$

$$\Rightarrow d\vec{M} = dx\, \hat{x} + dy\, \hat{y}$$

$$= (dr\cos\theta - r\sin\theta\, d\theta)(\cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta)$$
$$+ (dr\sin\theta + r\cos\theta\, d\theta)(\sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta)$$

$$= \hat{e}_r\, dr + r\, d\theta\, \hat{e}_\theta .$$

$u^1 \equiv r$, $u^2 \equiv \theta$ $\quad \begin{cases} \vec{e}_1 = \hat{e}_r \\ \vec{e}_2 = r\, \hat{e}_\theta \end{cases}$ local basis. $\begin{cases} du^1 = dr \\ du^2 = d\theta \end{cases}$

$$\Rightarrow d\vec{M} = du^i\, \vec{e}_i = \partial_i \vec{M}\, du^i$$

$$d\vec{M} \cdot d\vec{M} = \| d\vec{M} \|^2 = g_{ij}\, du^i du^j \; ; \; g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

$$g_{11} = \vec{e}_1 \cdot \vec{e}_1 = \hat{e}_r \cdot \hat{e}_r = 1 \; ; \; g_{22} = \vec{e}_2 \cdot \vec{e}_2 = r^2$$

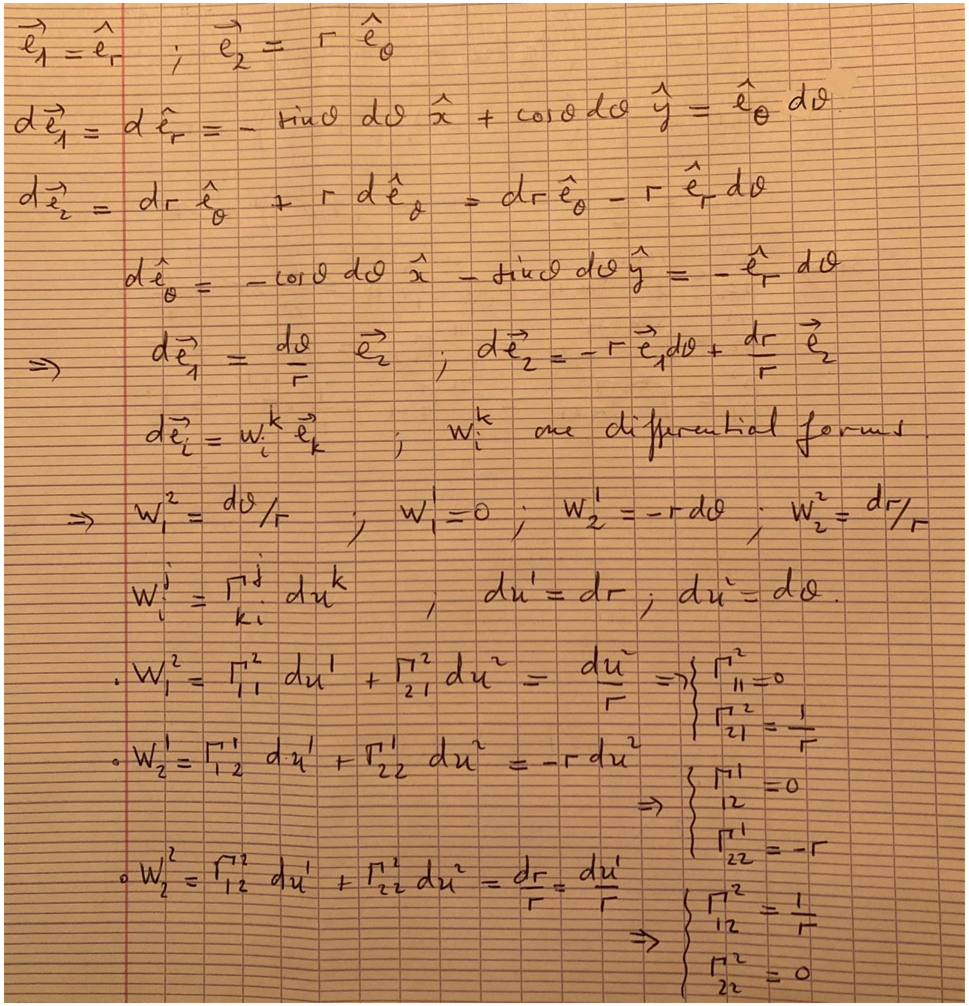
$$\Rightarrow g_{ij} : \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \; ; \; g_{ik} g^{kj} = \delta_i^j \; ;$$

- $g^{ii} = 1/g_{ii} \Rightarrow g^{11} = 1 \; ; \; g^{12} = g^{21} = 0 \; ; \; g^{22} = 1/r^2$

Exercices

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right) , \tag{33}$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) . \tag{34}$$



$\vec{e}_1 = \hat{e}_r \; ; \; \vec{e}_2 = r\,\hat{e}_\theta$

$d\vec{e}_1 = d\hat{e}_r = -\sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y} = \hat{e}_\theta\, d\theta$

$d\vec{e}_2 = dr\,\hat{e}_\theta + r\, d\hat{e}_\theta = dr\,\hat{e}_\theta - r\,\hat{e}_r\, d\theta$

$d\hat{e}_\theta = -\cos\theta\, d\theta\, \hat{x} - \sin\theta\, d\theta\, \hat{y} = -\hat{e}_r\, d\theta$

$\Rightarrow \; d\vec{e}_1 = \frac{d\theta}{r}\vec{e}_2 \; ; \; d\vec{e}_2 = -r\,\vec{e}_1\, d\theta + \frac{dr}{r}\vec{e}_2$

$d\vec{e}_i = w_i^k\, \vec{e}_k$; w_i^k are differential forms

$\Rightarrow \; w_1^2 = d\theta/r \; ; \; w_1^1 = 0 \; ; \; w_2^1 = -r\, d\theta \; ; \; w_2^2 = dr/r$

$w_i^j = \Gamma^j_{ki}\, du^k \; ; \; du^1 = dr \; ; \; du^2 = d\theta$.

- $W_1^2 = \Gamma^2_{11}\, du^1 + \Gamma^2_{21}\, du^2 = \frac{du^2}{r} \Rightarrow \begin{cases} \Gamma^2_{11} = 0 \\ \Gamma^2_{21} = \frac{1}{r} \end{cases}$
- $W_2^1 = \Gamma^1_{12}\, du^1 + \Gamma^1_{22}\, du^2 = -r\, du^2 \Rightarrow \begin{cases} \Gamma^1_{12} = 0 \\ \Gamma^1_{22} = -r \end{cases}$
- $W_2^2 = \Gamma^2_{12}\, du^1 + \Gamma^2_{22}\, du^2 = \frac{dr}{r} = \frac{du^1}{r} \Rightarrow \begin{cases} \Gamma^2_{12} = \frac{1}{r} \\ \Gamma^2_{22} = 0 \end{cases}$

$$R^i_{ljk} \rightarrow R_{jk} \rightarrow R$$

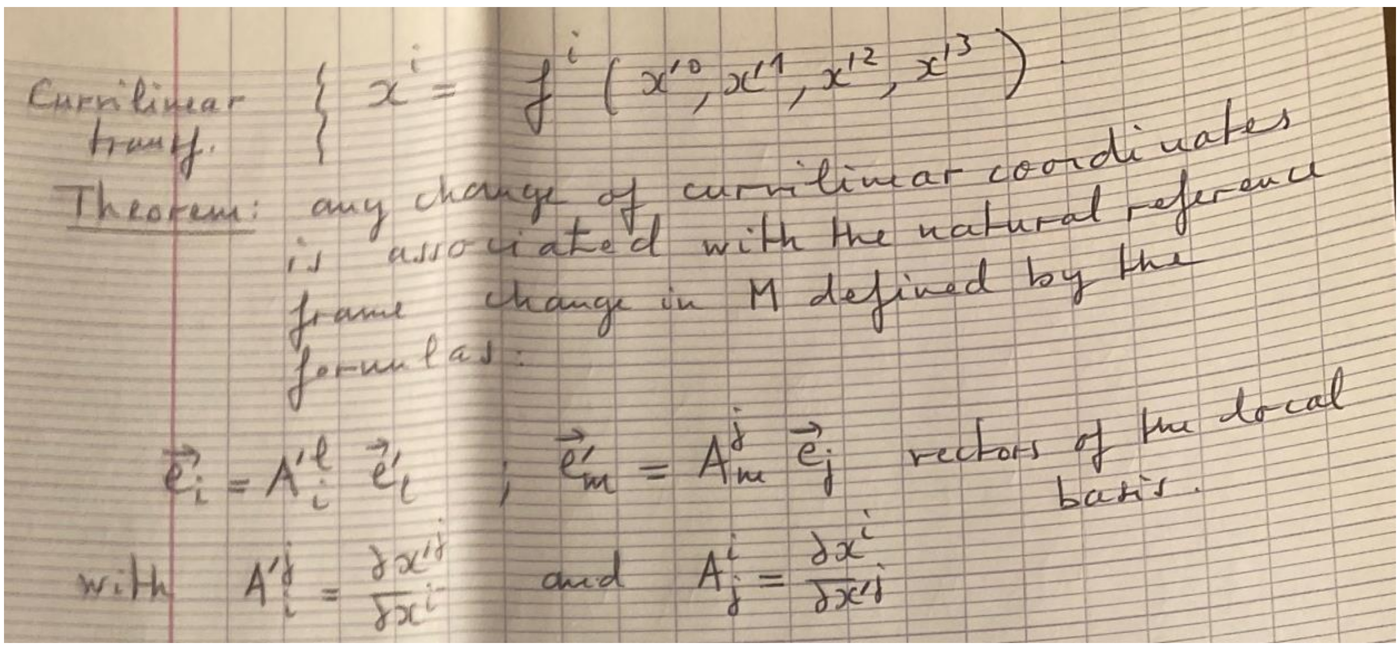
→ Conclusion?

Exercices

IV. EX4

Prove the transformation formula for Christoffel symbols:

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m} \; .$$



Curvilinear transf. $\{ x^i = f^i(x'^0, x'^1, x'^2, x'^3)$.

Theorem: any change of curvilinear coordinates is associated with the natural reference frame change in M defined by the formulas:

$\vec{e}_i = A'^l_i \vec{e}'_l$; $\vec{e}'_m = A^j_m \vec{e}_j$ vectors of the local basis.

with $A'^j_i = \frac{\partial x'^j}{\partial x^i}$ and $A^i_j = \frac{\partial x^i}{\partial x'^j}$

Exercices

$$d\vec{e}_i = A'^{\ell}_{i}\, d\vec{e}'_{\ell} + dA'^{\ell}_{i}\, \vec{e}'_{\ell} \qquad (0)$$

$$d\vec{e}_i = \omega^{j}_{i}\, \vec{e}_j = w^{j}_{i}\, \vec{e}_j \quad ; \quad d\vec{e}'_{\ell} = \omega'^{m}_{\ell}\, \vec{e}'_m .$$

$$(0) \Rightarrow \omega^{j}_{i}\, \vec{e}_j = A'^{\ell}_{i}\, \omega'^{m}_{\ell}\, \vec{e}'_m + dA'^{\ell}_{i}\, \vec{e}'_{\ell}$$

$$= A'^{\ell}_{i}\, A^{j}_{m}\, \omega'^{m}_{\ell}\, \vec{e}_j + A^{j}_{\ell}\, dA'^{\ell}_{i}\, \vec{e}_j$$

$$= \left(A'^{\ell}_{i}\, A^{j}_{m}\, \omega'^{m}_{\ell} + A^{j}_{\ell}\, dA'^{\ell}_{i} \right) \vec{e}_j$$

$$\Rightarrow \quad \omega^{j}_{i} = A'^{\ell}_{i}\, A^{j}_{m}\, \omega'^{m}_{\ell} + A^{j}_{\ell}\, dA'^{\ell}_{i}$$

$$\omega^{j}_{i} = \Gamma^{j}_{ki}\, dy^k \quad ; \quad \omega'^{m}_{\ell} = \Gamma'^{m}_{n\ell}\, dy'^n \quad ; \quad dy'^n = A'^{n}_{k}\, dy^k$$

dy^k: arbitrary numerical quantities

$$\Rightarrow \quad \Gamma^{j}_{ki} = A'^{\ell}_{i}\, A^{j}_{m}\, A'^{n}_{k}\, \Gamma'^{m}_{n\ell} + A^{j}_{\ell}\, \partial_k A'^{\ell}_{i}$$

$$\Rightarrow \quad \boxed{\Gamma^{j}_{ki} = \frac{\partial x'^{\ell}}{\partial x^i}\, \frac{\partial x^j}{\partial x'^m}\, \frac{\partial x'^n}{\partial x^k}\, \Gamma'^{m}_{n\ell} + \left\{ \frac{\partial^2}{\partial x^k \partial x^i}\, x'^{\ell} \right\} \frac{\partial x^j}{\partial x'^{\ell}}}$$

Exercices

V. EX5

The spherical coordinates are defined by: $(x = r\sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = r\cos\theta)$.

1) • Calculate the line element ds^2.

2) • Obtain the components of the metric tensor g_{ij}.

3) • Calculate the Christoffel symbols of the first kind in spherical coordinates.

4) • Calculate those of the second kind.

1) Note the coordinates $u^1 = r,\ u^2 = \theta,\ u^3 = \varphi$

$d\vec{M} = du^i\vec{e}_i$ with $\vec{e}_1 \equiv \hat{e}_r$, $\vec{e}_2 \equiv r\hat{e}_\theta$ and $\vec{e}_3 \equiv r\sin\theta\hat{e}_\varphi$ a local basis.

We have: $d\vec{M}\cdot d\vec{M} = ds^2 = g_{ij}du^i du^j$ with $g_{ij} = \vec{e}_i\cdot\vec{e}_j$.

2) It leads to: $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2\sin^2\theta$ and $g_{ij} = 0$ if $i \neq j$.

3) We then have: $\partial_1 g_{22} = 2r$, $\partial_1 g_{33} = 2r\sin^2\theta$ and $\partial_2 g_{33} = 2r^2\cos\theta\sin\theta$.

Exercices

Applying the formula

$$\Gamma_{kji} = \frac{1}{2}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}\right) \ ,$$

we obtain

$$\Gamma_{212} = -\, r \quad ; \quad \Gamma_{323} = -r^2 \sin\theta\cos\theta \quad ; \quad \Gamma_{313} = -r\sin^2\theta$$
$$\Gamma_{122} = \Gamma_{221} = r \quad ; \quad \Gamma_{133} = \Gamma_{331} = r\sin^2\theta \quad ; \quad \Gamma_{332} = \Gamma_{233} = r^2\cos\theta\sin\theta$$

4) Using $\Gamma_{k\ j}^{\ i} = g^{il}\,\Gamma_{klj}$ and $g_{ik}g^{kj} = \delta_i^j$

we obtain

$$\Gamma_{2\ 2}^{\ 1} = -\, r \quad ; \quad \Gamma_{2\ 1}^{\ 2} = \Gamma_{1\ 2}^{\ 2} = 1/r \quad ; \quad \Gamma_{3\ 1}^{\ 3} = \Gamma_{1\ 3}^{\ 3} = 1/r$$
$$\Gamma_{3\ 3}^{\ 1} = -\, r\sin^2\theta \quad ; \quad \Gamma_{3\ 3}^{\ 2} = -\sin\theta\cos\theta \quad ; \quad \Gamma_{2\ 3}^{\ 3} = \Gamma_{3\ 2}^{\ 3} = \mathrm{cotan}\,\theta$$

$$R^i_{ljk} \to R_{jk} \to R$$

→ Conclusion?

Exercices

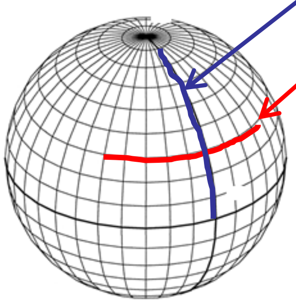
VI. EX6

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0$$

A particle moves along a trajectory defined in spherical coordinates (r, θ, φ). Determine the contravariant components a^k of the acceleration $\vec{a}$ of this particle for the following trajectories:

1) • The trajectory is defined by: $r = c, \theta = \omega t, \varphi = \pi/4$ where t is the time.

2) • The trajectory is defined by: $r = c, \theta = \pi/4, \varphi = \omega t$. Calculate the norm of the acceleration and show that we find the back classic formula: $\|\vec{a}\| = r\omega^2$.



Exercices

1) Let's determine the values of the Christoffel symbols along the trajectory;

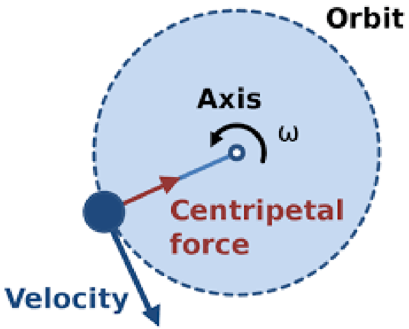
We have for $r = c,\ \theta = \omega\, t,\ \varphi = \pi/4$

$$\Gamma_{2\ 1}^{\ 2} = \Gamma_{1\ 2}^{\ 2} = \frac{1}{u^1} = \frac{1}{c} \quad ; \quad \Gamma_{3\ 1}^{\ 3} = \Gamma_{1\ 3}^{\ 3} = \frac{1}{u^1} = \frac{1}{c} \quad ; \quad \Gamma_{3\ 2}^{\ 3} = \Gamma_{2\ 3}^{\ 3} = \mathrm{cotan}\,\omega\, t$$

$$\Gamma_{2\ 2}^{\ 1} = -u^1 = -c \quad ; \quad \Gamma_{3\ 3}^{\ 1} = -c\sin^2\omega\, t \quad ; \quad \Gamma_{3\ 3}^{\ 2} = -\sin\omega\, t\cos\omega\, t$$

The contravariant components of acceleration are as follows:

$$a^1 = \frac{d^2\, u^1}{\mathrm{d}t^2} + \Gamma_{i\ k}^{\ 1}\frac{\mathrm{d}u^i}{\mathrm{d}t}\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0 + \Gamma_{2\ 2}^{\ 1}\left(\frac{\mathrm{d}u^2}{\mathrm{d}t}\right)^2 + \Gamma_{3\ 3}^{\ 1}\left(\frac{\mathrm{d}u^3}{\mathrm{d}t}\right)^2 = -c\,\omega^2$$



$$a^2 = 0 \quad ; \quad a^3 = 0$$

We find the classical expression for the acceleration of a particle travelling a circular trajectory at constant speed.

$$u^1 = r,\ u^2 = \theta,\ u^3 = \varphi$$

$$\boxed{\frac{d^2u^i}{dt^2} + \Gamma^i_{kl}\frac{du^k}{dt}\frac{du^l}{dt}}$$

Exercices

2) Christoffel symbols along the trajectory:

$$\Gamma_2{}^2{}_1 = \Gamma_1{}^2{}_2 = \frac{1}{u^1} = \frac{1}{c} \;\; ; \;\; \Gamma_3{}^3{}_1 = \Gamma_1{}^3{}_3 = \frac{1}{u^1} = \frac{1}{c} \;\; ; \;\; \Gamma_3{}^3{}_2 = \Gamma_2{}^3{}_3 = \mathrm{cotan}\,\pi/4 = 1$$

$$\Gamma_2{}^1{}_2 = -u^1 = -c \;\; ; \;\; \Gamma_3{}^1{}_3 = -c\sin^2\pi/4 = -(c/2) \;\; ; \;\; \Gamma_3{}^2{}_3 = -(1/2)$$

The contravariant components of acceleration are as follows:

$$a^1 = \frac{d^2\,u^1}{\mathrm{d}t^2} + \Gamma_i{}^1{}_k\,\frac{\mathrm{d}u^i}{\mathrm{d}t}\,\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0 + \Gamma_2{}^1{}_2\left(\frac{\mathrm{d}u^2}{\mathrm{d}t}\right)^2 + \Gamma_3{}^1{}_3\left(\frac{\mathrm{d}u^3}{\mathrm{d}t}\right)^2 = -\frac{c\,\omega^2}{2}$$

$$a^2 = 0 + 2\,\Gamma_1{}^2{}_2\,\frac{\mathrm{d}u^1}{\mathrm{d}t}\,\frac{\mathrm{d}u^2}{\mathrm{d}t} + \Gamma_3{}^2{}_3\left(\frac{\mathrm{d}u^3}{\mathrm{d}t}\right)^2 = -\frac{\omega^2}{2} \;\; ; \;\; a^3 = 0$$

The covariant components of the fundamental tensor along the trajectory are :

$$g_{11} = 1 \;\; ; \;\; g_{22} = c^2 \;\; ; \;\; g_{33} = c^2/2 \;\; \text{d'où} \;\; ||\mathbf{a}|| = \sqrt{g_{ij}\,a^i\,a^j} = c\,\omega^2/\sqrt{2}$$

The radius of the circle covered is $r = c\sin(\pi/4) = c/\sqrt{2}$ leading to

$$||\mathbf{a}|| = c\,\omega^2/\sqrt{2} = r\,\omega^2$$