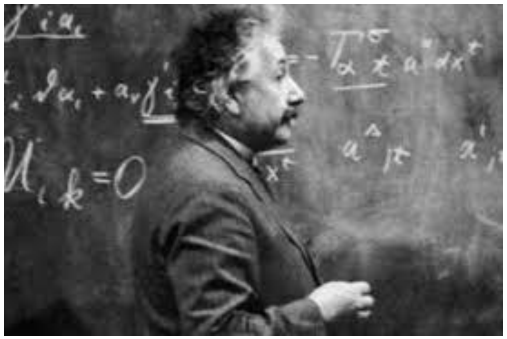
General Relativity (GR)

M1 - Physique 2023-2024



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$$g_{\mu\nu}$$

AE+GR (1907-1917)

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R-2\lambda)d^4x$$

IV) Applications

- The Newtonian limit
- Time dilation in GR
- Gravitational field created by a massive object
- Gravitational waves
- Black holes
- Cosmology

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

The Newtonian limit

Let us examine how, in the limiting case, the metric tensor g_{ik} determining the field is related to the non relativistic potential ϕ of the gravitational field.

The motion of a particle in a gravitational field ϕ is determined, in NR mechanics by a lagrangian having (in an inertial reference frame) the form

$$L = \frac{mv^2}{2} - m\phi \, , \tag{5}$$

where $\phi(\vec{r}, t)$ characterizes the field and is called the gravitational potential. The equation of motion (Euler-Lagrange) of the particle is

$$\dot{\vec{v}} = -\vec{\nabla}\phi \, , \tag{6}$$

Let us write the lagrangian (5) in the form

$$L = -mc^2 + \frac{mv^2}{2} - m\phi \, . \tag{7}$$

The Newtonian limit

Note: $L = -mc^2 + \frac{mv^2}{2}$ shall be the same exactly as that to which the corresponding relativistic function $L = -mc^2\sqrt{1-\beta^2}$ reduces in the limit $\beta \to 0$.

The NR action S has the form

$$S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt \; . \tag{8}$$

Comparing this with the expression $S = -mc \int ds$ we see that

$$ds = \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt \; , \tag{9}$$

leading to

$$ds^2 = \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right)^2 dt^2 = (c^2 + 2\phi) dt^2 - dr^2 \; , \tag{10}$$

where we have neglected the terms with $1/c^2$ and used $\vec{v}dt = d\vec{r}$, $dr^2 = d\vec{r} \cdot d\vec{r}$.

Therefore the component g_{00} of the metric tensor is given by

$$g_{00} = 1 + \frac{2\phi}{c^2} \; . \tag{11}$$

The Newtonian limit

Further we can use the expression of the stress-energy tensor:

$$T_i^k = \mu_0 c^2 \bar{u}_i \bar{u}^k \ , \tag{12}$$

where μ_0 is the *rest* mass density of the body.

As for the four-velocity $\bar{u}^i$, since the macroscopic motion is also considered to be slow we must neglect all its space components and retain only the time component. So we have: $\bar{u}^\alpha = 0$ with $\alpha = 1..3$ and $\bar{u}^0 = \bar{u}_0 = 1$. It leads to

$$T_0^0 \equiv T = \mu_0 c^2 \ . \tag{13}$$

Using the field equations

$$R_i^k = \frac{8\pi G}{c^2}\left(T_i^k - \frac{1}{2}\delta_i^k T\right) \ , \tag{14}$$

we get

$$R_0^0 = \frac{4\pi G \mu_0}{c^2} \ . \tag{15}$$

The Newtonian limit

From $R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$ and using the fact that:

$$\begin{cases} \Gamma \propto \dfrac{\partial g}{\partial x_0} & (16) \\ \dfrac{\partial \Gamma}{\partial x^0} \ll \dfrac{\partial \Gamma}{\partial x^\alpha} & (17) \\ \Gamma^l_{ik}\Gamma^m_{lm} \propto 1/c^2 & (18) \\ \Gamma^m_{il}\Gamma^l_{km} \propto 1/c^2 & (19) \end{cases}$$

we get

$$R_{00} = R^0_0 = \frac{\partial \Gamma^\alpha_{00}}{\partial x^\alpha} \ . \tag{20}$$

From $\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$ we obtain

$$\Gamma^\alpha_{00} = \frac{1}{2} g^{\alpha m} \left(-\frac{\partial g_{00}}{\partial x^m} \right) \tag{21}$$

$$= \frac{1}{2} g^{\alpha\alpha} \left(-\frac{\partial g_{00}}{\partial x^\alpha} \right) \tag{22}$$

$$= \frac{1}{c^2} \frac{\partial \phi}{\partial x^\alpha} \ . \tag{23}$$

Therefore $R^0_0 = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^{\alpha 2}} \equiv \frac{1}{c^2} \Delta \phi$ leading to

$$\boxed{\Delta \phi = 4\pi G \mu_0 \ .} \tag{24}$$

This is the Poisson equation where m_0 is the source term and ϕ the field.

The Newtonian limit

The integral solution of this equation reads as

$$\phi(\vec{r}) = -G \int \frac{\mu_0(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \ . \tag{25}$$

If $\mu_0(\vec{r}') = m_0 \delta(\vec{r}')$ then $\phi(\vec{r}) = -\frac{Gm_0}{|\vec{r}|}$. The correction to the Minkowski metric is

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{26}$$

$$= \left(1 - \frac{2Gm_0}{c^2 r}\right) c^2 dt^2 - dx^2 - dy^2 - dz^2 \ . \tag{27}$$

On the earth's surface we have:

$$\begin{cases} m_0 = M_\oplus = 5.972 \times 10^{24} \ \ \mathrm{kg} \\ c = 3 \times 10^8 \ \ \mathrm{m/s} \\ r = r_\oplus = 6371 \ \ \mathrm{km} \\ \Rightarrow \frac{2GM_\oplus}{c^2 r_\oplus} \simeq 1.3 \times 10^{-9} \ . \end{cases} \tag{28}$$

Consequently, the correction to the Minkowski metric in our environment (the Earth's surface) is of the order of a billionth. As the equations show, this is enough to cause bodies to fall! This corresponds to the geodesics of the slightly modified metric. Remember that for a Minkowski space (with the metric $ds^2 = c^2 dt^2 - dx^2$... and $R_{00} = 0$) there is NO GRAVITATION!

General Relativity (GR)

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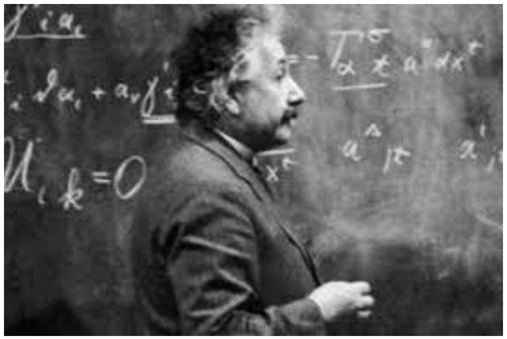

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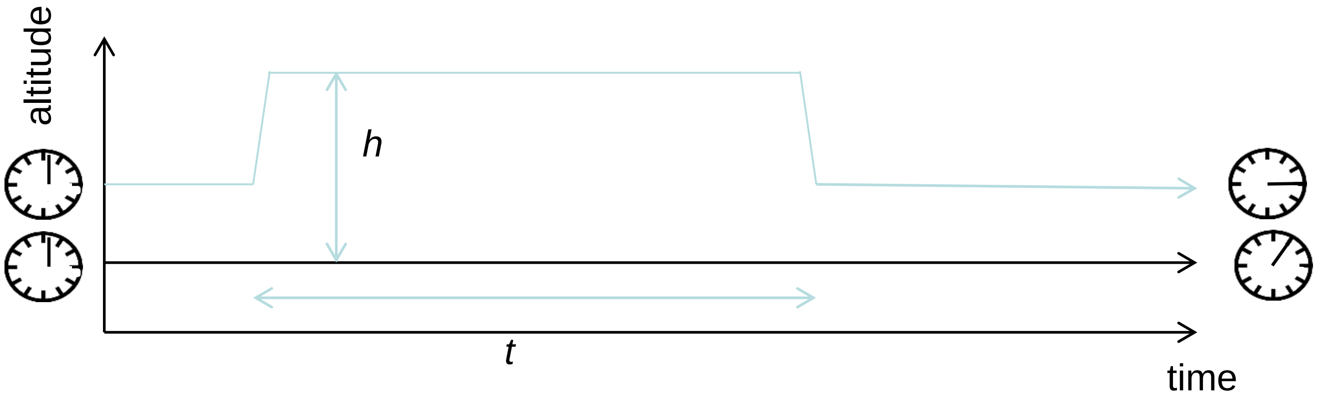
Time dilation

This is a truly new prediction of the theory.

At the Earth's surface, the gravitational potential is $\phi = gh$ where h is the height above ground. We have (*)

$$ds^2 = \left(1 + \frac{2gh}{c^2}\right) c^2dt^2 - dx^2 - dy^2 - dz^2 \ . \quad (1)$$

Let's consider two equal clocks. One is kept on the ground $(z = 0)$ and the second is set at altitude h for a value t of the time coordinate. Then bring it back to ground level and compare the two clocks' indications.



(*) Valid if $h \ll r_{\oplus}$

Time dilation

At time t, the time T_{down} measured by the ground clock is t because at $h = 0$ the metric is that of Minkowski ($ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$). But this is not the case for the upper clock: it indicates the proper time, i.e.

$$T_{up} = \frac{1}{c}\int_0^t \sqrt{g_{00}dx^0dx^0} = \frac{1}{c}\int_0^t \sqrt{\left(1+\frac{2gh}{c^2}\right)c^2dt^2} \sim \left(1+\frac{gh}{c^2}\right)t > T_{down}\ . \qquad (2)$$

This result is quite spectacular: a clock runs faster if it is at a higher altitude. The relative difference between the measured times is given by

$$\boxed{\frac{\Delta T}{T} = \frac{T_{up} - T_{down}}{T_{down}} = \frac{gh}{c^2}\ .} \qquad (3)$$

For $h = 1$ m this gives

$$\boxed{\frac{\Delta T}{T} \sim 10^{-16}\ .} \qquad (4)$$

This means that if the clock is kept 1 meter higher than another for 100 days ($\sim 10^7$ s), the lower clock will lag behind it by 1 ns (10^{-9} s).

Time dilation

- Today's best clocks have an accuracy of over 10^{-16} s, and this effect has been extensively verified in the laboratory.
- Clocks run more slowly if they are lower in the gravitational potential.
- Note: A red shift (we move towards the infrared) indicates a decrease in light energy (we move towards the infrared) which, using Planck's $E = \hbar\omega = \frac{2\pi\hbar}{T}$ formula, corresponds to an increase in period.
- The satellites of the American GPS system orbit at a radius of $R \simeq 26600$ km. Calculate the time it would have taken for the system to accumulate a 3 km location error on Earth if it had been set up before the discovery of general relativity, and therefore ignoring the effects of the latter.

GPS signals travel at the speed of light. They therefore cover a length of $l = 3$ km in a time of $\Delta T = l/c \simeq 10^{-5}$ s. The relativistic correction is as follows:

$$\Delta T/T = \frac{GM}{r_{\oplus}c^2} - \frac{GM}{Rc^2} \simeq 10^{-9} \, .$$

So $T = \Delta T/10^{-9} \simeq 10^4$ s: less than three hours.

Understanding general relativity has therefore played a key role in the construction of GPS navigation systems.

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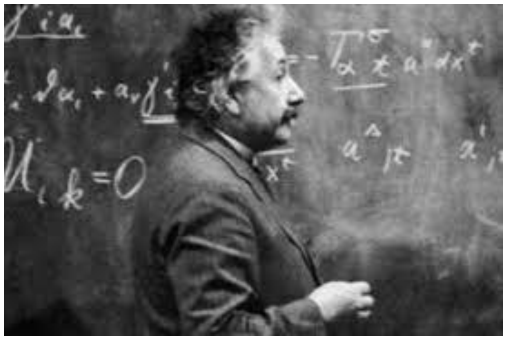

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Gravitational field created by a massive object

The Schwarzschild line element

- This is an exact solution of Einstein's equations for the limited case of a single spherical non-rotating mass.
- First obtained by **Karl Schwarzschild**, German physicist and astronomer.
- He obtained this result in 1915, the same year that Einstein first introduced general relativity.
- His last articles were published in 1916.

(1873 – **11 May 1916**)

first paper

On the Gravitational Field of a Point-Mass, According to Einstein's Theory

Karl Schwarzschild

Submitted on January 13, 1916

Abstract: This is a translation of the paper *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie* by Karl Schwarzschild, where he obtained the metric of a space due to the gravitational field of a point-mass. The paper was originally published in 1916, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, S. 189–196. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski.

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second paper

On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein's Theory

Karl Schwarzschild

Submitted on February 24, 1916

Abstract: This is a translation of the paper *Über das Gravitationsfeld einer Kugel aus incompressiebler Flüssigkeit nach der Einsteinschen Theorie* published by Karl Schwarzschild, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 424–435. Here Schwarzschild expounds his previously obtained metric for the spherically symmetric gravitational field produced by a point-mass, to the case where the source of the field is represented by a sphere of incompressible fluid. Schwarzschild formulates the physical condition of degeneration of such a field. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski.

Gravitational field created by a massive object

The Schwarzschild line element

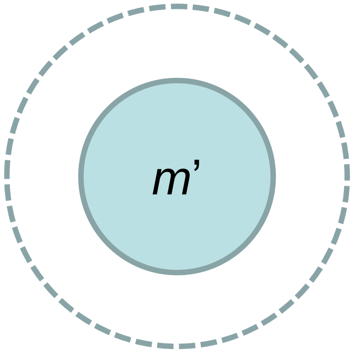
The Schwarzschild metric is as follows

$$ds^2 = (1 - r_g/r)\, c^2 dt^2 - r^2(\sin^2\theta d^2\varphi + d^2\theta) - \frac{dr^2}{1 - r_g/r}\,, \quad (29)$$

where $r_g = \frac{2Gm'}{c^2}$ is the gravitational radius (please check the physical dimension).

$$x^i = (x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \varphi)$$

- From (29) one can compute the Christoffel symbols Γ^k_{ij} and then the Ricci tensor R^i_k.
- We assume that $\Lambda = 0$ in the Einstein equation.
- In order to solve the Einstein's equation one needs to specify the stress-energy tensor T^j_i. One can use the general form $T_{ik} = (p + \varepsilon)\bar{u}_i\bar{u}_k - pg_{ik}$.
- In the case of a symmetrical central field in vacuum, i.e. outside the masses generating it, we have $T^k_i = 0$ and the problem is completely integrable. This is the so-called Schwarzschild solution.



Gravitational field created by a massive object

Directly from (29) we have: $g_{00} = 1 - r_g/r$, $g_{11} = -\left(\frac{1}{r - r_g/r}\right)$, $g_{22} = -r^2$, and $g_{33} = -r^2 \sin^2\theta$. It is easy to show that: $g^{00} = (1 - r_g/r)^{-1}$, $g^{11} = -\left(\frac{1}{r - r_g/r}\right)^{-1} = -(1 - r_g/r)$, $g^{22} = -r^{-2}$, and $g^{33} = -r^{-2}\sin^{-2}\theta$.

Using $\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$ we obtain:

$$\begin{cases} \Gamma^2_{12} = \Gamma^3_{13} = 1/r \\ \Gamma^3_{23} = \cot\theta \\ \Gamma^1_{33} = -r\sin^2\theta\,(1 - r_g/r) \\ \Gamma^1_{22} = -r\,(1 - r_g/r) \\ \Gamma^2_{33} = -\sin\theta\cos\theta \Gamma^1_{11} = \lambda'/2 \; \Gamma^0_{10} = \nu'/2 \\ \Gamma^0_{11} = \frac{\dot\lambda}{2} e^{\lambda-\nu} = 0 \\ \Gamma^1_{00} = \frac{\nu'}{2} e^{\nu-\lambda} \\ \Gamma^0_{00} = \frac{\dot\nu}{2} = 0 \\ \Gamma^1_{10} = \frac{\dot\lambda}{2} = 0\,, \end{cases} \tag{30}$$

with

$$\begin{cases} e^{-\lambda} = e^{\nu} = 1 - r_g/r \\ b' \equiv \frac{\partial}{\partial r} \\ \dot a \equiv \frac{\partial}{\partial (ct)}\,. \end{cases} \tag{31}$$

Gravitational field created by a massive object

Since λ and ν do not depend on t we have $\dot{\lambda} = \dot{\nu} = 0$. Moreover $\lambda = \ln\left(\frac{1}{1-r_g/r}\right)$.

From the Einstein's equation we verify that $R_{ik} = 0$. Please check using the definition:

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} \ . \tag{32}$$

The equation of motion is

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0 \tag{33}$$

Using $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$ and $ds = cd\tau$ we have for example $i = 0$:

$$\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0 \ , \tag{34}$$

and for the other components

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta e^{-\lambda}\left(\frac{d\varphi}{ds}\right)^2 + \frac{e^{\nu-\lambda}}{2}\frac{d\nu}{dr}\left(\frac{dt}{ds}\right)^2 = 0 \tag{35}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = 0 \tag{36}$$

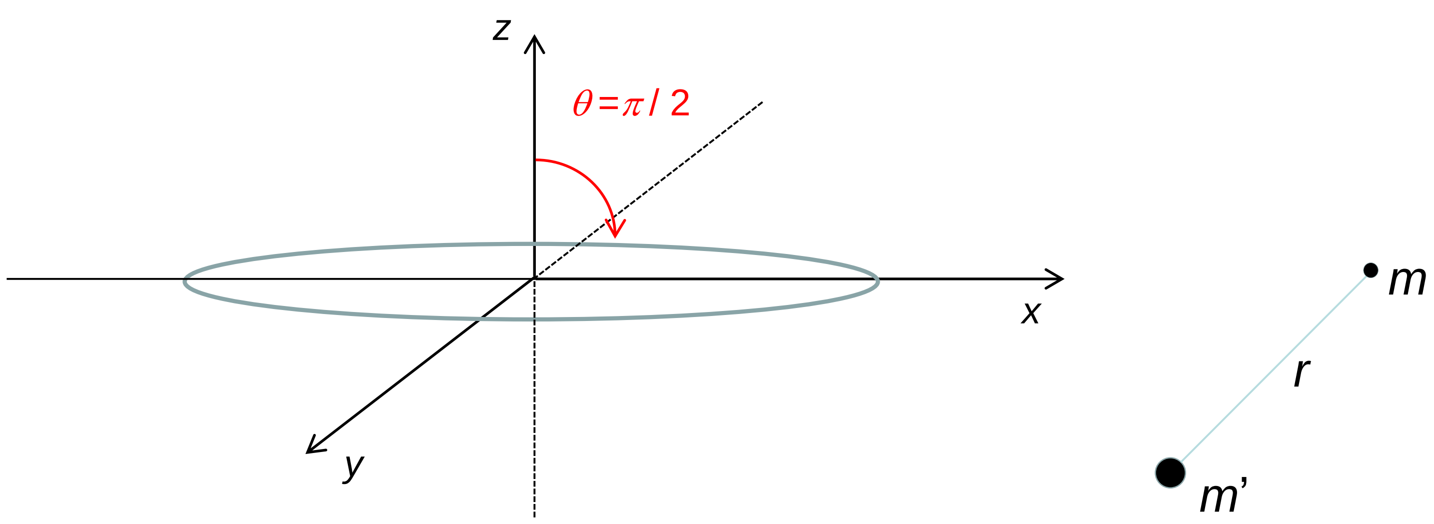
$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\cot\theta\frac{d\varphi}{ds}\frac{d\theta}{ds} = 0 \tag{37}$$

$$\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0 \ . \tag{38}$$

Gravitational field created by a massive object

By choosing $\theta = \pi/2$ corresponding to a trajectory in the plan (xoy) it leads to $\cos\theta = \cot\theta = 0$ and $d^2\theta/ds^2 = d\theta/ds = 0$. The new equations of motion are as follows

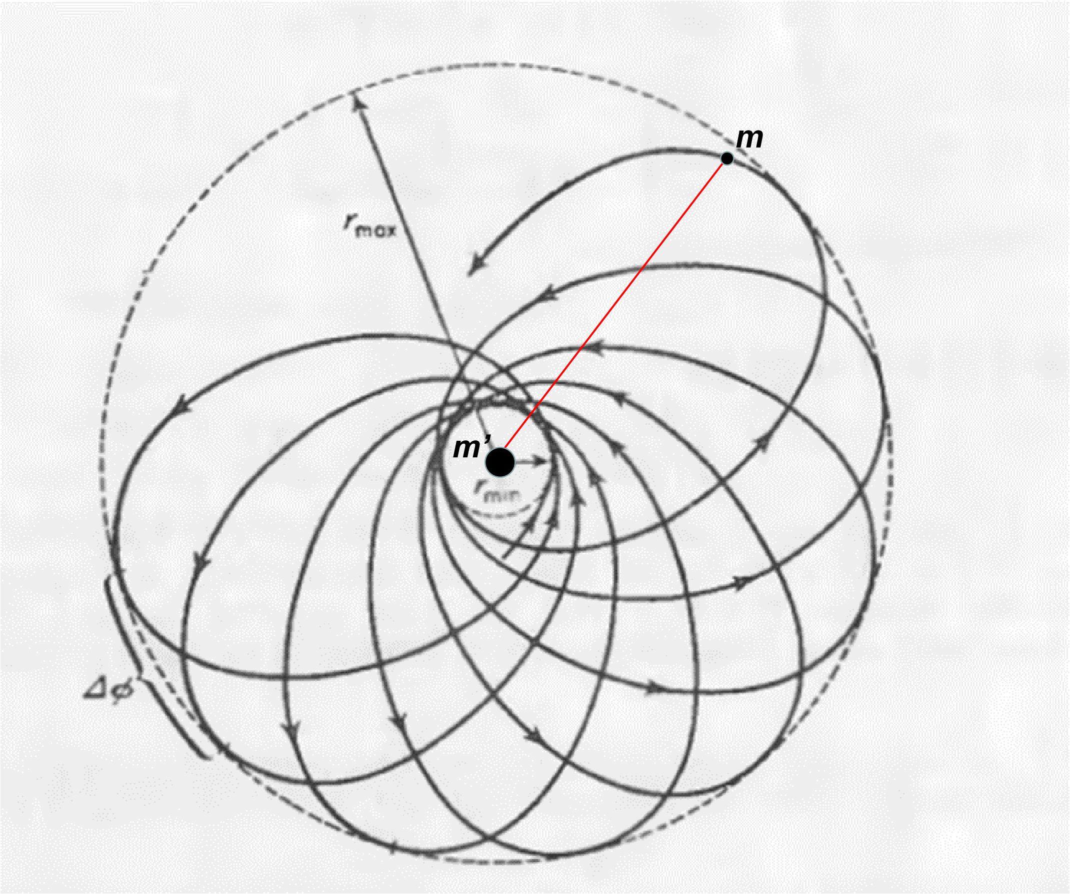
$$\begin{cases} \dfrac{d^2r}{ds^2} + \dfrac{1}{2}\dfrac{d\lambda}{dr}\left(\dfrac{dr}{ds}\right)^2 - re^{-\lambda}\left(\dfrac{d\varphi}{ds}\right)^2 + \dfrac{e^{\nu-\lambda}}{2}\dfrac{d\nu}{dr}\left(\dfrac{dt}{ds}\right)^2 = 0 & (39) \\ \dfrac{d^2\varphi}{ds^2} + \dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\varphi}{ds} = 0 & (40) \\ \dfrac{d^2t}{ds^2} + \dfrac{d\nu}{dr}\dfrac{dr}{ds}\dfrac{dt}{ds} = 0\ . & (41) \end{cases}$$

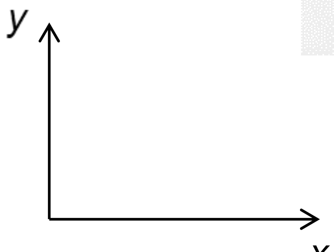


xoy : plane of the trajectory

Gravitational field created by a massive object

$$\Delta\phi = \Delta\varphi$$





Gravitational field created by a massive object

- The equation $\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = 0$ leads to $\frac{d\varphi}{ds} = \frac{h}{r^2}$ with h a constant of integration.

- The equation $\frac{d^2t}{ds^2} + \frac{\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0$ leads to $\frac{dt}{ds} = ke^{-\nu}$ with k a constant of integration.

From $ds^2 = (1 - r_g/r)\, c^2dt^2 - r^2(\sin^2\theta d^2\varphi + d^2\theta) - \frac{dr^2}{1-r_g/r}$ one gets

$$(1 - r_g/r)^{-1}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 - (1 - r_g/r)c^2\left(\frac{dt}{ds}\right)^2 - 1 = 0\ . \qquad (42)$$

Using $1 - r_g/r \equiv e^\nu$ and $\frac{dt}{ds} = ke^{-\nu}$ we obtain

$$\boxed{\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{r_gc^2}{r}\left(1 + \frac{r^2}{c^2}\left(\frac{d\varphi}{d\tau}\right)^2\right) = \mathrm{cte}\ .} \qquad (43)$$

This differential equation must be compared with the NR result:

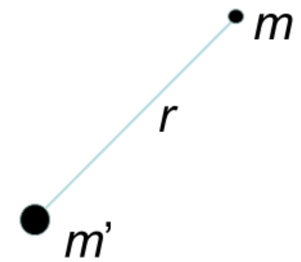
$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 + \frac{2}{m}U(r) = \mathrm{cte}\ (= 2E/m)\ , \qquad (44)$$

with $U(r) = -\frac{Gmm'}{r} \equiv -\frac{\alpha}{r}$ leading to

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 - \frac{2Gm'}{r} = \mathrm{cte}\ . \qquad (45)$$

Gravitational field created by a massive object

The equations (45) and (43) are identical if:



1. on neglects the second member in parenthesis in (43)
2. $v << c \Rightarrow \gamma = 1 \Rightarrow d\tau = dt$
3. we set $r_g \equiv 2\frac{Gm'}{c^2}$

Using $L \equiv mr^2\frac{d\varphi}{dt}$ one can rewrite (45) as

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{Gmm'}{r} + \frac{L^2}{2mr^2} - \frac{Gm'L^2}{c^2m}\frac{1}{r^3} = \text{cte} \ . \qquad (46)$$

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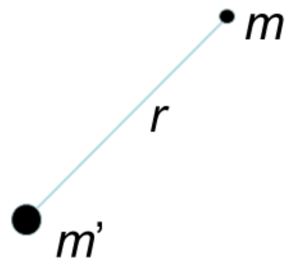

Constant of motion: orbital angular momentum (fixed by IC)

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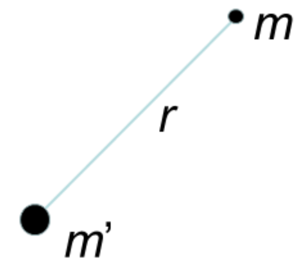

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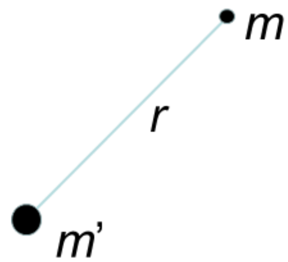
Usual gravitational potential energy (attractive force, Newton)

Gravitational field created by a massive object



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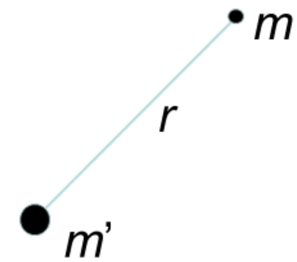
Using $L \equiv mr^2\frac{d\varphi}{dt}$ one can rewrite (45) as

Centrifugal barrier (repulsive force)

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{Gmm'}{r} + \frac{L^2}{2mr^2} - \frac{Gm'L^2}{c^2m}\frac{1}{r^3} = \text{cte} \; . \qquad (46)$$

Gravitational field created by a massive object

The equations (45) and (43) are identical if:



1. on neglects the second member in parenthesis in (43)
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Supplementary potential energy (attractive force, RG)

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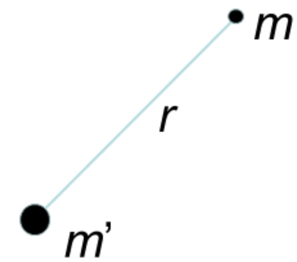

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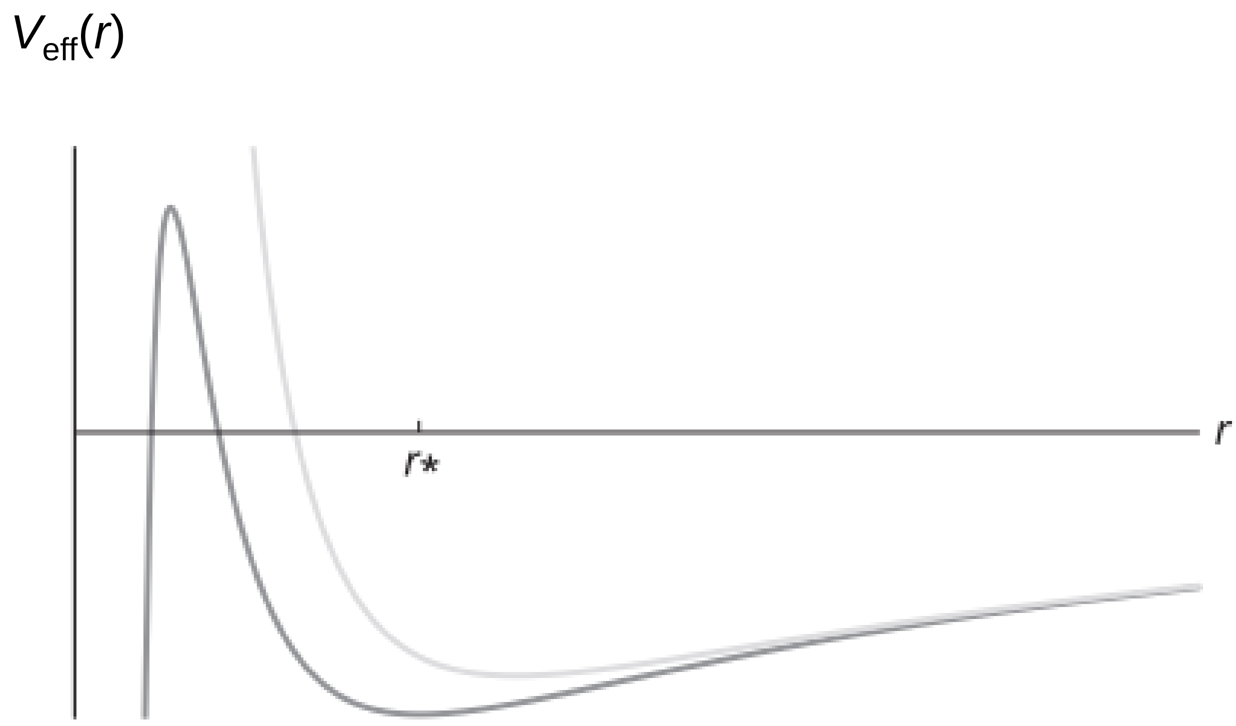
Usual gravitational potential energy (attractive force, Newton)

Supplementary potential energy (attractive force, RG)

- The quantity $-\frac{Gm'L^2}{c^2m}\frac{1}{r^3} \equiv \gamma/r^3 \equiv \delta U$ has the physical dimension of an energy (please check).

Gravitational field created by a massive object

$$V_{\text{eff}}(r) = -\frac{Gmm'}{r} + \frac{L^2}{2mr^2} - \frac{Gm'L^2}{c^2 m}\frac{1}{r^3}$$



The effective potential $V_{\text{eff}}(r)$ for a massive particle around a central mass.
The curve further to the right is the Newtonian one.

Gravitational field created by a massive object

The relativistic effect of gravity on an object orbiting a central mass *m'* is simply this additional attractive force

$$F = -\frac{3Gm'L^2}{c^2 m}\frac{1}{r^4}$$

- First, it is proportional to L^2, namely to the radial velocity. This means that it is a magnetic-like force: it is not felt by a mass without angular velocity.
- Second, it is inversely proportional to c^2; therefore, it is a relativistic effect and is small for non-relativistic velocities.
- Third, it is proportional to r^{-4}, which means that it becomes important – in fact, dominant – at small radii. In the solar system, the planet with the smallest radius and the largest angular velocity is Mercury; therefore, we may expect that Mercury is the first planet where the effect of this relativistic force has a chance to be detected.
- Fourth, it is proportional to m^{-1}. Therefore the smaller the mass of the body in orbit around *m'*, the greater the force.

Gravitational field created by a massive object

According to the textbook "Mechanics" by Landau we have $\delta\varphi = -\frac{6\pi\alpha\gamma m^2}{L^4}$. Using $a(1-e^2) = \frac{L^2}{am^2m'G}$ where e is the eccentricity of the orbit, one gets the final expression

$$\boxed{\delta\varphi = \frac{6\pi G m'}{c^2 a(1-e^2)}\text{rad} .} \tag{47}$$

For Mercury we have $a = 57.91 \times 10^6$ km and $e = 0.206$. Using $G = 6.674 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} and $m' = 1.989 \times 10^{30}$ kg (solar mass) on get's (2π rad = 360 deg and 1 deg = 3600 arcsecond)

$$\delta\varphi = 0.103'' \quad \text{for one period} . \tag{48}$$

Given that we have 415 revolutions per century we have

$$\boxed{\delta\varphi = 42.8'' \quad \text{per century} .} \tag{49}$$

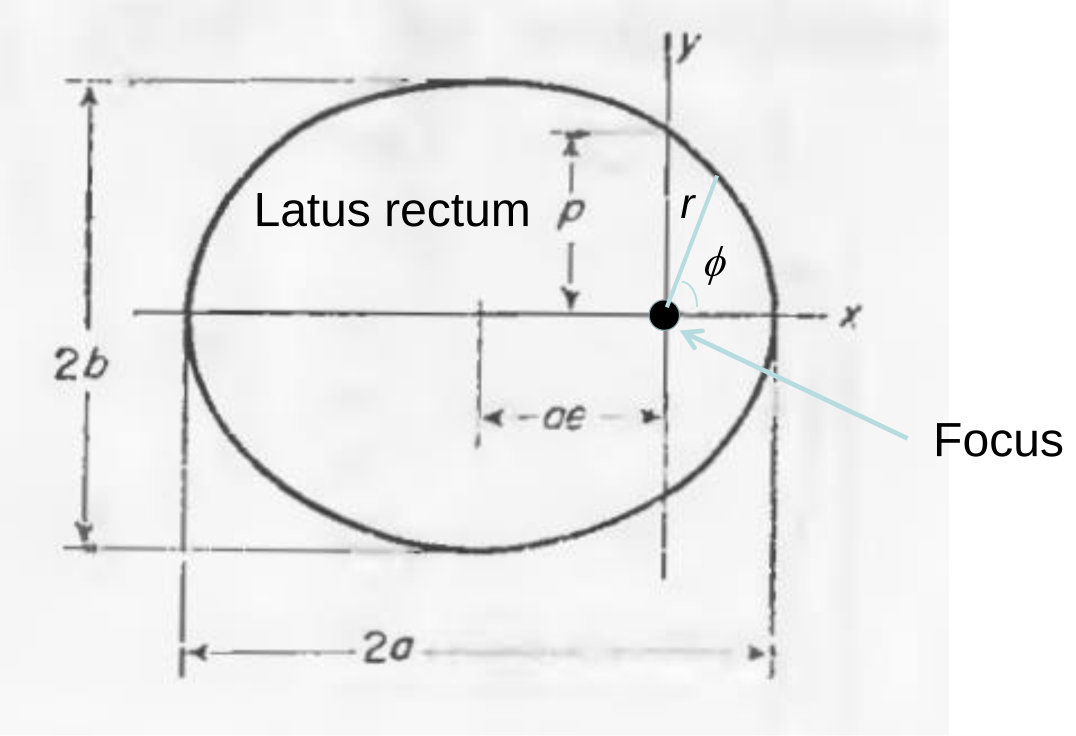
This theoretical prediction is in excellent agreement with the measurements which is

$$\boxed{\delta\varphi_{\text{measured}} = 42'' \pm 1'' \quad \text{per century} .} \tag{50}$$

Gravitational field created by a massive object

$$U = -\alpha/r$$

$$p = M^2/m\alpha,$$



$$p/r = 1 + e\cos\phi.$$

Polar equation

$$r_{\min} = p/(1+e) = a(1-e)$$

$$r_{\max} = p/(1-e) = a(1+e)$$

Gravitational field created by a massive object

$$\begin{cases} \Delta\phi = \Delta\varphi \\ M = L \\ U(r) = -\frac{Gmm'}{r} - \frac{Gm'L^2}{c^2 m}\frac{1}{r^3} \end{cases}$$

$$\Delta\phi = 2\int_{r_{\min}}^{r_{\max}} \frac{M\,\mathrm{d}r/r^2}{\sqrt{[2m(E-U)-M^2/r^2]}}. \qquad (14.10)$$

PROBLEM 3. When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, the paths of finite motion are no longer closed, and at each revolution the perihelion is displaced through a small angle $\delta\phi$. Find $\delta\phi$ when (a) $\delta U = \beta/r^2$, (b) $\delta U = \gamma/r^3$.

SOLUTION. When r varies from $r_{\min}$ to $r_{\max}$ and back, the angle ϕ varies by an amount (14.10), which we write as

$$\Delta\phi = -2\frac{\partial}{\partial M}\int_{r_{\min}}^{r_{\max}} \sqrt{\left[2m(E-U)-\frac{M^2}{r^2}\right]}\,\mathrm{d}r,$$

in order to avoid the occurrence of spurious divergences. We put $U = -\alpha/r + \delta U$, and expand the integrand in powers of δU; the zero-order term in the expansion gives 2π, and the first-order term gives the required change $\delta\phi$:

$$\delta\phi = \frac{\partial}{\partial M}\int_{r_{\min}}^{r_{\max}} \frac{2m\delta U\,\mathrm{d}r}{\sqrt{\left[2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}\right]}} = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} r^2\delta U\,\mathrm{d}\phi\right), \qquad (1)$$

where we have changed from the integration over r to one over ϕ, along the path of the "unperturbed" motion.

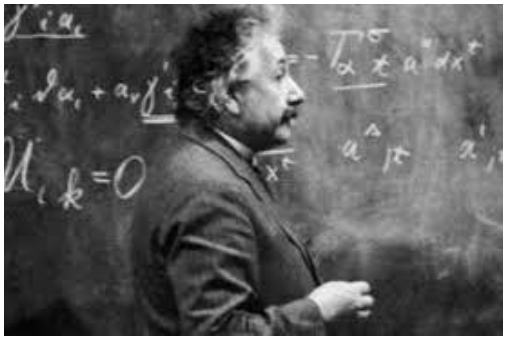
In case (a), the integration in (1) is trivial: $\delta\phi = -2\pi\beta m/M^2 = -2\pi\beta/\alpha p$, where $2p$ (15.4) is the latus rectum of the unperturbed ellipse. In case (b) $r^2\delta U = \gamma/r$ and, with $1/r$ given by (15.5), we have $\delta\phi = -6\pi\alpha\gamma m^2/M^4 = -6\pi\gamma/\alpha p^2$.

Gravitational field created by a massive object

Note (Rovelli): ”In his long search for field equations, Einstein recalculated this precession several times, with each provisional formulation of his field equations. When he found the field equations leading to the correct value (not using the Schwarzschild metric, which he didn’t yet have, but the approximate solution $ds^2 = -(1 - M/r)dt^2 + (1 + 2M/r)dr^2 + r^2 d\Omega^2$), he convinced himself they were the right ones.”

General Relativity (GR)

M1 - Physique 2023-2024



Paul-Antoine Hervieux
Unistra/IPCMS
hervieux@unistra.fr

$$g_{\mu\nu}$$

AE+GR (1907-1917)

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

IV) Applications

- The Newtonian limit
- Time dilation in GR
- Gravitational field created by a massive object
- The gravitational deflexion of light
- Black holes
- Cosmology

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

The gravitational deflexion of light

One can rewrite $ds^2 = (1 - r_g/r)\, c^2 dt^2 - r^2(\sin^2\theta d^2\varphi + d^2\theta) - \frac{dr^2}{1 - r_g/r}$ as

$$ds^2 = \gamma dt^2 - r^2(\sin^2\theta d^2\varphi + d^2\theta) - \frac{c^2}{\gamma} dr^2 \ , \tag{52}$$

with $\gamma = (1 - r_g/r)\, c^2$.

Dividing by ds^2 the above expression one gets

$$\frac{c^2}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 - \gamma\left(\frac{dt}{ds}\right)^2 = -1 \ . \tag{53}$$

We have seen that:

$$r^2 \frac{d\varphi}{ds} = h \tag{54}$$

$$\frac{dt}{ds} = k e^{-\nu} = \frac{k}{\gamma} \ , \tag{55}$$

where h and k are two constants. Posing $u = 1/r$ and using the above three equations we obtain

$$\boxed{u + \frac{d^2 u}{d\varphi^2} = \frac{r_g}{2h^2} + \frac{3 r_g}{2} u^2 \ .} \tag{56}$$

The gravitational deflexion of light

To describe the propagation of light, we must impose $ds = 0$. Using $r^2 \frac{d\varphi}{ds} = h$, we can see that this is equivalent to making h tend towards infinity $h \to \infty$. Equation (55) becomes

$$\frac{d^2u}{d\varphi^2} + u = \frac{3r_g}{2} u^2 \ . \tag{57}$$

This second order differential equation can be integrated by successive approximations.

The solution of $\frac{d^2u}{d\varphi^2} + u = 0$ is $u = \frac{1}{R} \cos\varphi$. This leads to a new differential equation, which is an approximation of the one we want to solve

$$\frac{d^2u}{d\varphi^2} + u = \frac{3r_g}{2} \frac{1}{R^2} \cos^2\varphi \ . \tag{58}$$

Please check that $u_1 = \frac{r_g}{2R^2}\left(\cos^2\varphi + 2\sin^2\varphi\right)$ is a particular solution of (57). Therefore one can write

$$\boxed{u = \frac{1}{R}\cos\varphi + \frac{r_g}{2R^2}\left(\cos^2\varphi + 2\sin^2\varphi\right) \equiv \frac{1}{r} \ .} \tag{59}$$

The gravitational deflexion of light

By using the cartesian coordinates $(x = r\cos\varphi, y = r\sin\varphi)$ (58) can be rewritten as

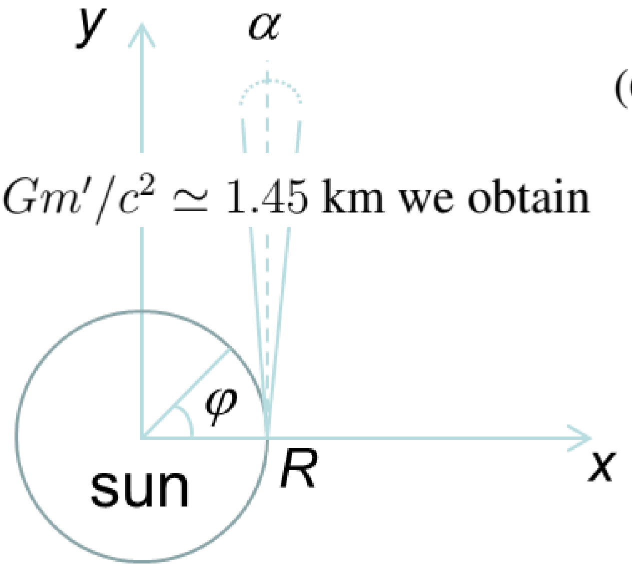
$$x = R - \left(\frac{r_g}{2R}\right)\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}} \,. \tag{60}$$

The second term on the right-hand side expresses the slight deviation of the light ray from the line $x = R$ (see graph). We have

$$y \gg x \Rightarrow x = R \pm \frac{r_g}{R} y \,. \tag{61}$$

The angle of the asymptotes, i.e. the total deviation of the light as it passes through the gravitational field, is therefore

$$\boxed{\alpha = \frac{2r_g}{R} \,.} \tag{62}$$



Using $R =$ solar radius$=7 \times 10^5$ km, $r_g = 2Gm'/c^2$ with $Gm'/c^2 \simeq 1.45$ km we obtain

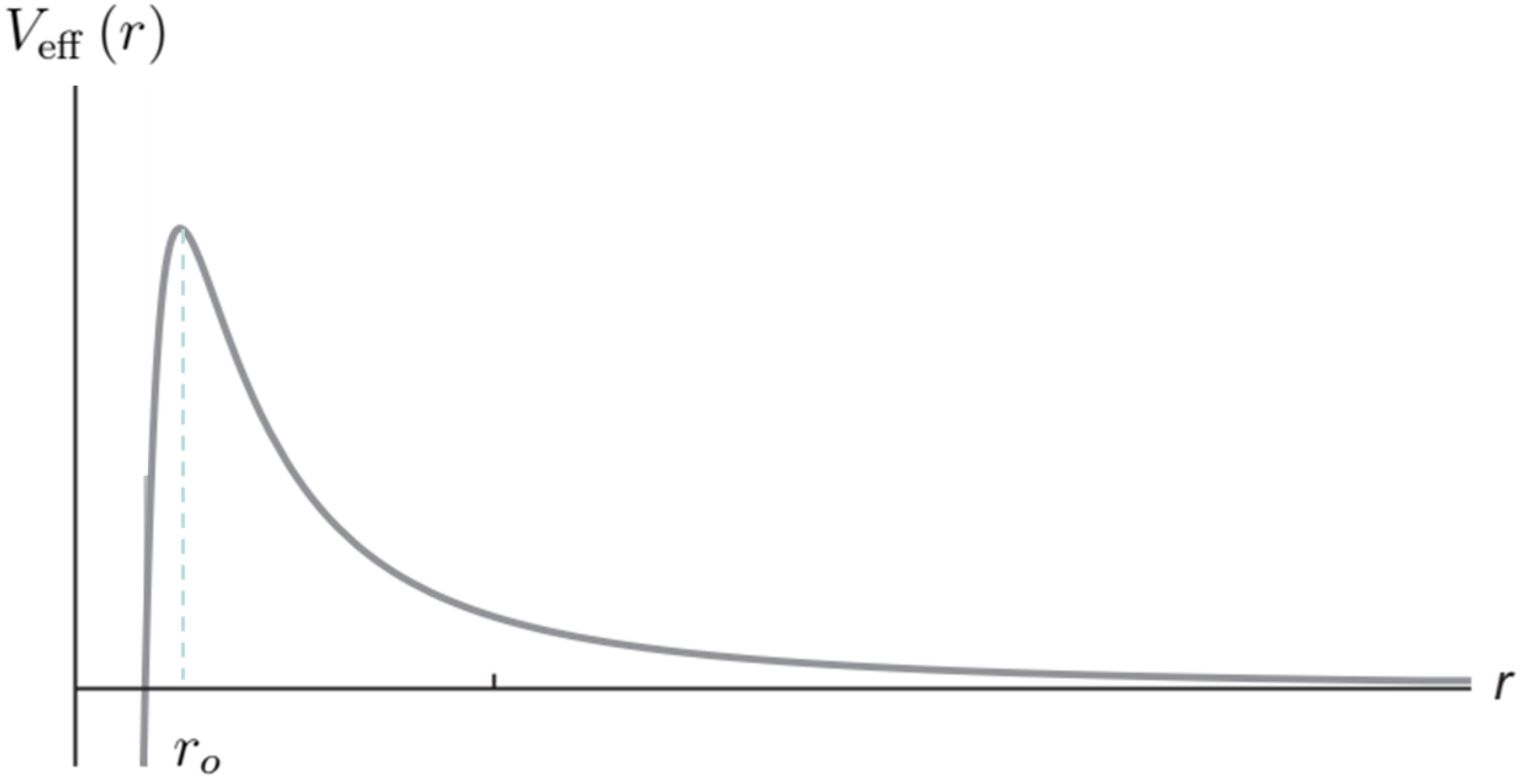
$$\alpha = \frac{4Gm'}{c^2 R} = 8.28 \times 10^{-6} \text{ rad} \simeq 1.71'' \,.$$

Can light go into orbit around a black hole?

To study the trajectory of light, we can use the effective potential:

$$V_{\text{eff}}(r) = \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3},$$

(see demonstration after)

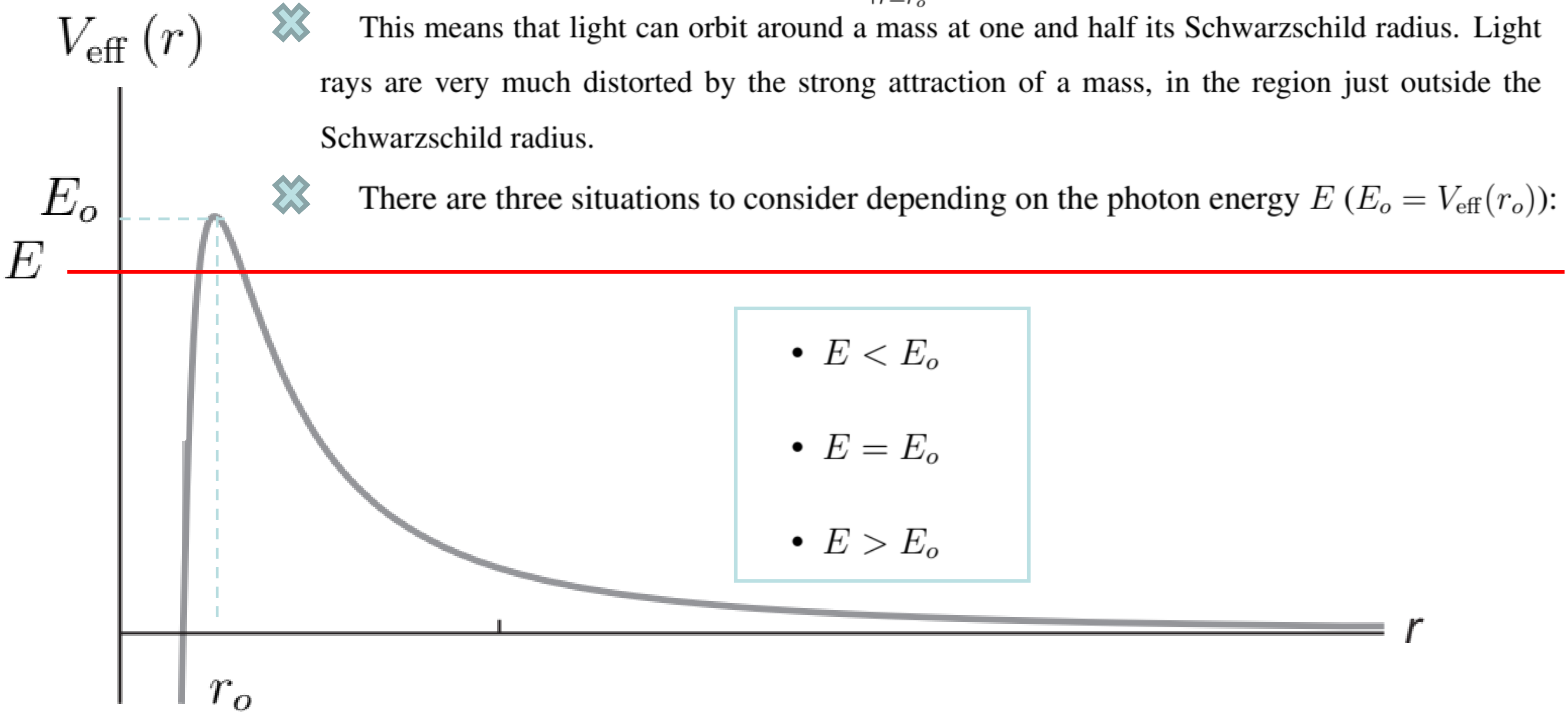


Can light go into orbit around a black hole?

The photon sphere is a region near a black hole where the gravity is so strong that light itself can orbit around the black hole. The orbits in that region are unstable; the photons can loop around the black hole a few times, but they will not stay forever.

The effective potential has a maximum at r_o such that $\partial_r V_{\text{eff}}(r)|_{r=r_o} = 0$ leading to $r_o = \frac{3}{2} r_g$.

- This position is instable because $\left.\frac{d^2 V_{\text{eff}}(r)}{dr^2}\right|_{r=r_o} < 0$.
- This means that light can orbit around a mass at one and half its Schwarzschild radius. Light rays are very much distorted by the strong attraction of a mass, in the region just outside the Schwarzschild radius.
- There are three situations to consider depending on the photon energy E ($E_o = V_{\text{eff}}(r_o)$):

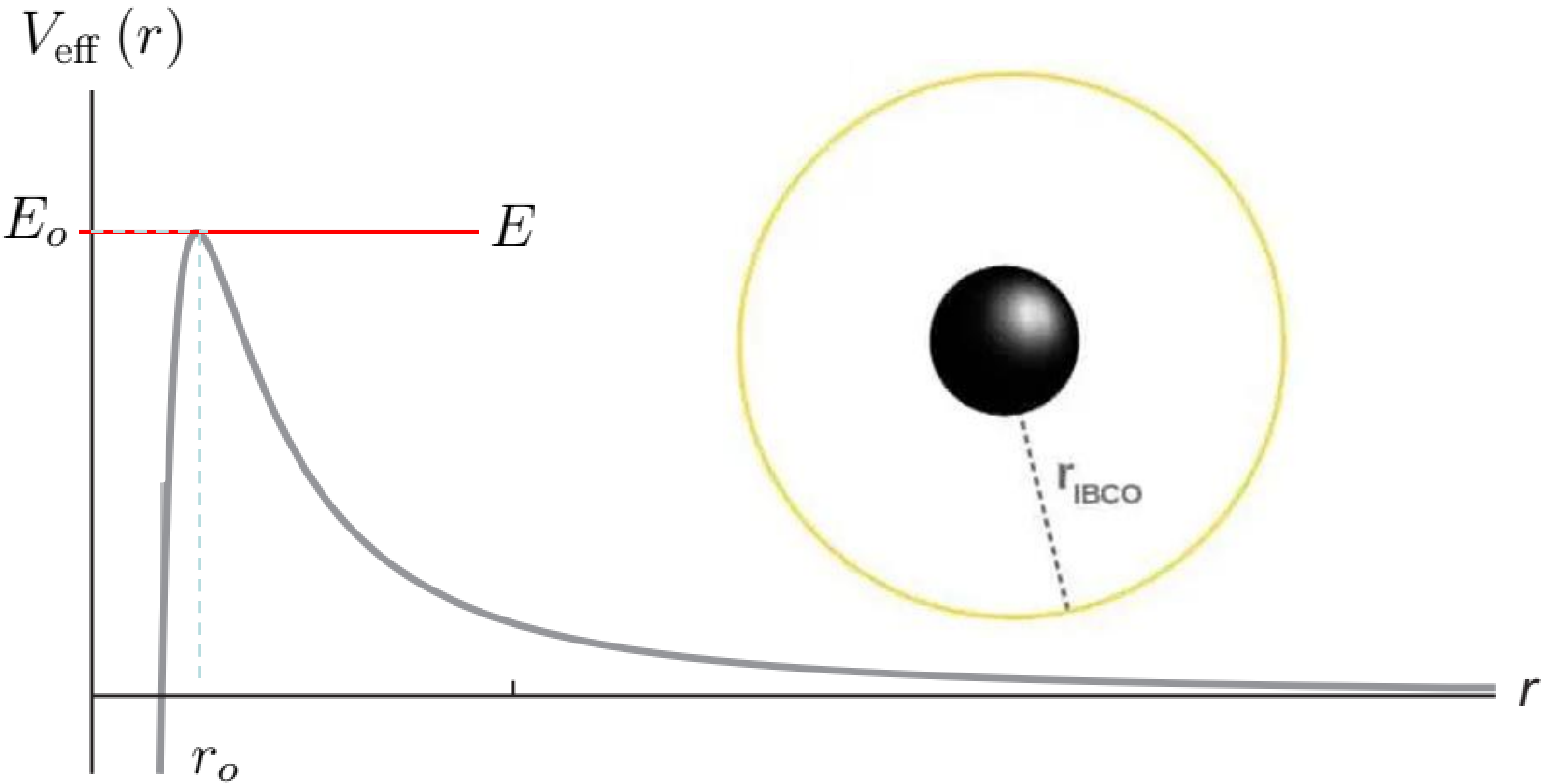


- $E < E_o$
- $E = E_o$
- $E > E_o$

Can light go into orbit around a black hole?

$$E = E_o$$

The photon stays in an **unstable** circular orbit, which defines the **Innermost Bound Circular Orbit** (IBCO) around the black hole. As only light can orbit at this radius, it is sometimes referred to as the **photon sphere**.

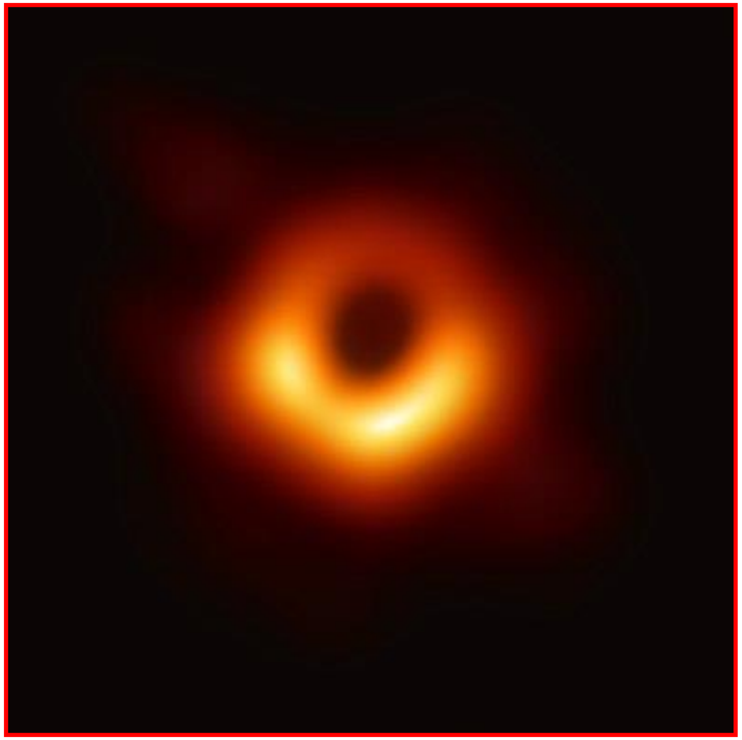


Can light go into orbit around a black hole?

- Messier 87 (also known as Virgo A or NGC 4486, generally abbreviated to M87) is a supergiant elliptical galaxy in the constellation Virgo.

- M87 is about 16.4 million parsecs (53 million light-years= 5×10^{20} km) from Earth.

- Astronomers Reveal First Image of the Black Hole **at the Heart of Our Galaxy**, May, EHT **2022.**



Direct image of a supermassive black hole at the core of M87. Event Horizon Telescope (EHT) Collaboration **2019**.

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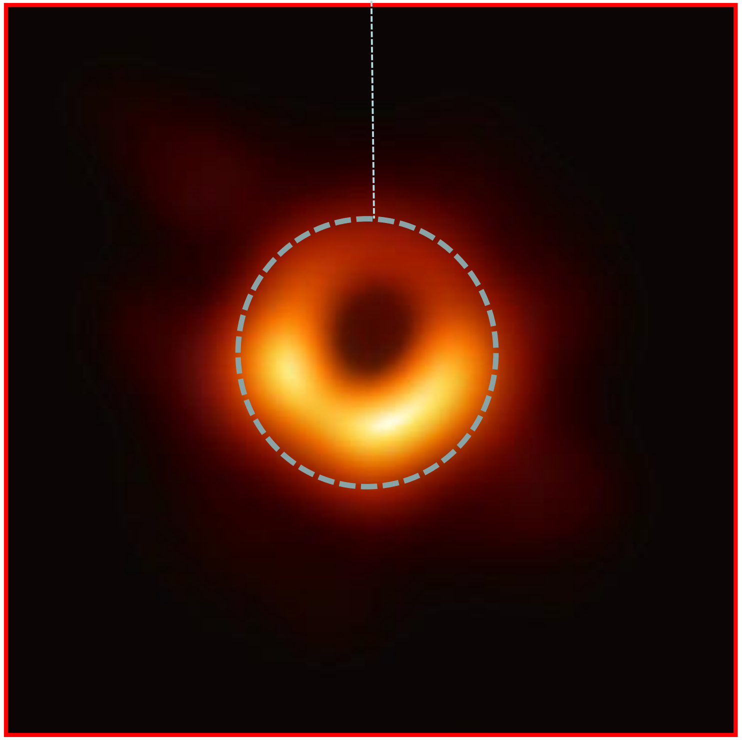
GR prediction r_{IBCO}



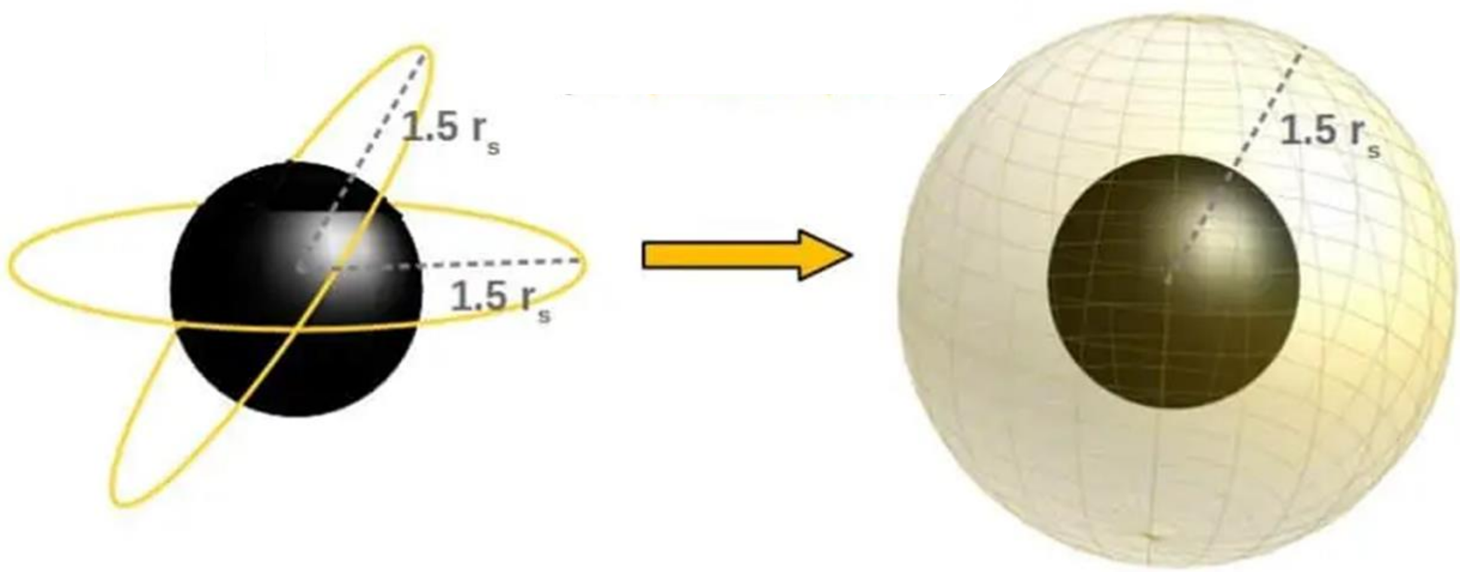
Direct image of a supermassive black hole at the core of M87. Event Horizon Telescope (EHT) Collaboration **2019**.

Can light go into orbit around a black hole?

$$E = E_o$$

A photon can have a circular orbit in **any orbital plane** around the black hole.

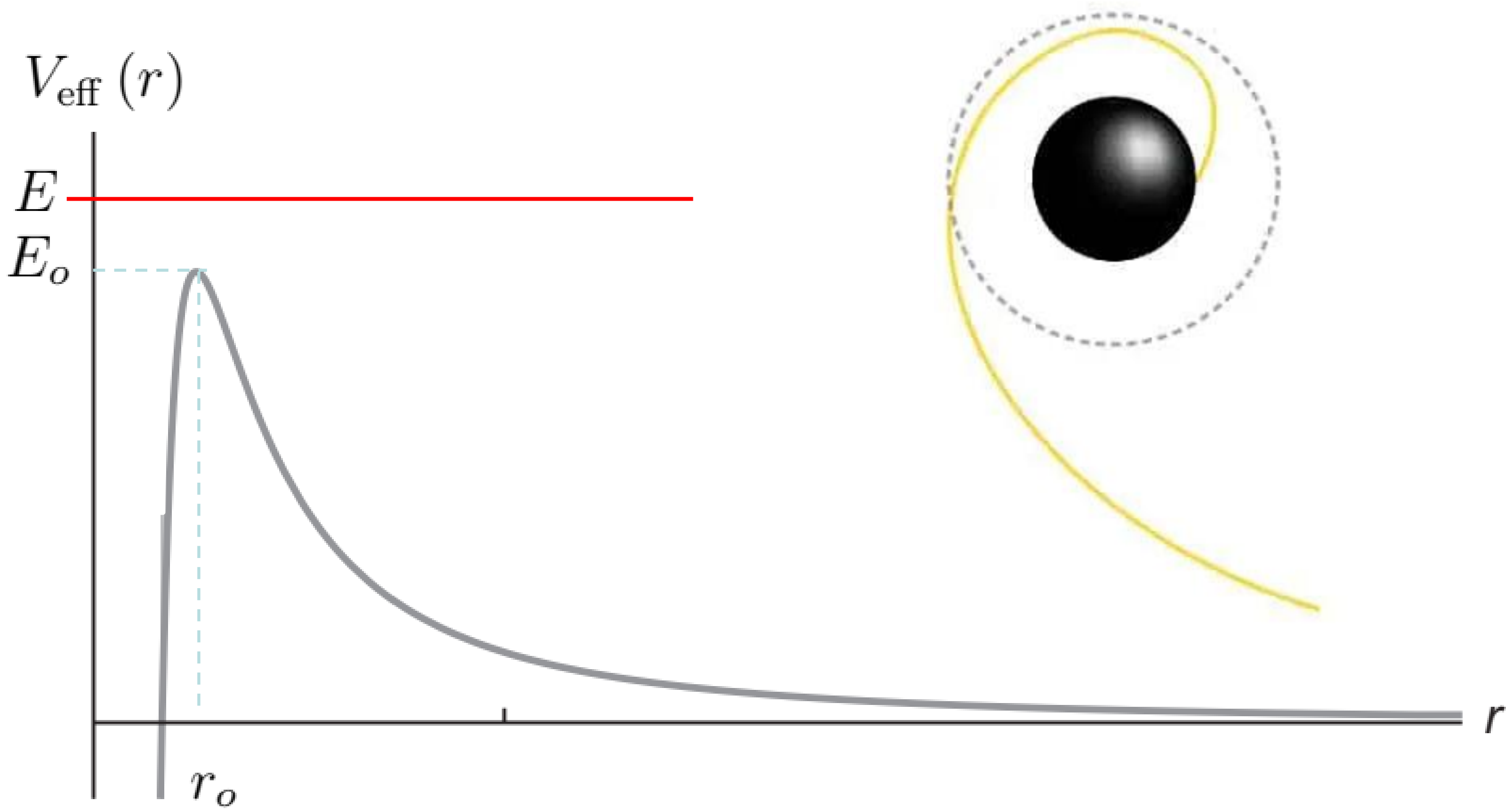
Together, all possible orbital planes **form a sphere of possible orbits of light** around the balck hole



Can light go into orbit around a black hole?

$$E > E_o$$

If the photon crosses the IBCO radius, **it will inevitably spiral into the black hole**.



Demonstration of V_{eff}

Our starting point will be the equation we derived earlier that describes the null geodesics of light around a black hole ($ds = 0; \theta = \pi/2; d^2\theta = 0$):

$$-\left(1 - \frac{r_g}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 d\varphi^2 = 0 \; . \tag{65}$$

From now on $c = 1$. We'll now divide both sides by an affine parameter $d\lambda^2$, which basically defines the rate of change of these space-time coordinates along the trajectory of the light ray, giving us:

$$-\left(1 - \frac{r_g}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{1 - \frac{r_g}{r}}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \; . \tag{66}$$

You can think of the affine parameter as playing the same role as time does in ordinary mechanics; it defines the rates of change of things like position, which then gives us the velocity. In relativity, we use an affine parameter instead of time since time itself is one of our space-time coordinates (the most commonly used affine parameter is called proper time, however, this is not defined for particles moving at the speed of light, so we cannot use it here).

Demonstration of V_{eff}

These $dt/d\lambda$, $dr/d\lambda$ and $d\varphi/d\lambda$ are essentially the space-time velocities or rates of change of our space-time coordinates. We'll denote these by putting a dot above the coordinate in question, so the equation then becomes:

$$-\left(1-\frac{r_g}{r}\right)\dot{t}^2+\frac{1}{1-\frac{r_g}{r}}\dot{r}^2+r^2\dot{\varphi}^2=0\ . \tag{67}$$

So far, what we've done may seem a little random. The goal with this is to end up with an equation that resembles the total energy of light in a familiar form. This will allows us to define the effective potential (since the total energy E will be of the form $E =$"radial" kinetic energy+effective potential).

Demonstration of V_{eff}

Also, we use here an extremely useful "trick" to find the Lagrangian instantly from any metric line element ds^2. We can then, using this method, instantly deduce that our Lagrangian will be:

$$L = -\frac{1}{2}\left(1 - \frac{r_g}{r}\right)\dot{t}^2 + \frac{1}{2}\frac{1}{1 - \frac{r_g}{r}}\dot{r}^2 + \frac{1}{2}r^2\dot{\varphi}^2 \, . \tag{68}$$

Notice the similarity between this and the line element formula. Essentially, the "trick" is that to find the Lagrangian, you have to just replace the coordinate displacements in the line element with these coordinate velocities (things with the dots above them) as well as divide everything by 2.

Now, using this Lagrangian we can notice two things; the Lagrangian only depends on the coordinate r, but not the coordinates t and φ (the Lagrangian does depend on t-dot and φ-dot, but not on t and φ themselves).

If you're familiar with Lagrangian mechanics, specifically Noether's theorem, this means that there exists conserved quantities associated with both of these coordinates (t and φ).

Demonstration of V_{eff}

We can derive these quantities by writing out the Euler-Lagrange equations for both of these coordinates:

$$\frac{\partial L}{\partial t} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) \tag{69}$$

$$\frac{\partial L}{\partial \varphi} = \left(\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\varphi}}\right) \quad . \tag{70}$$

From these equations, we get by inserting the Lagrangian (notice that the right-hand side on both of these is automatically zero as the Lagrangian does not depend on t or φ at all, and therefore $\partial L/\partial t = 0$ and $\partial L/\partial \varphi = 0$):

$$\frac{d}{d\lambda}\left(\left(1 - \frac{r_g}{r}\right)\dot{t}\right) = 0 \tag{71}$$

$$\frac{d}{d\lambda}\left(r^2\dot{\varphi}\right) = 0 \ . \tag{72}$$

Demonstration of V_{eff}

Now let's think about what it means for the derivative (with respect to the affine parameter λ in this case) of something to be zero as we have here; it means that the quantity has to be a constant.

So, we then get two constants of motion from the above equations (the first one is the energy E and the second is the orbital angular momentum L; these both arise from Noether's theorem), which allows us to express the coordinate velocities in terms of these constants:

$$\left(1-\frac{r_g}{r}\right)\dot{t}=E \quad \Rightarrow \quad \dot{t}=\frac{E}{\left(1-\frac{r_g}{r}\right)} \tag{73}$$

$$r^2\dot{\varphi}=L \quad \Rightarrow \quad \dot{\varphi}=\frac{L}{r^2} \; . \tag{74}$$

Now remember the equation for the null geodesics we had earlier:

$$-\left(1-\frac{r_g}{r}\right)\dot{t}^2+\frac{1}{1-\frac{r_g}{r}}\dot{r}^2+r^2\dot{\varphi}^2=0 \; . \tag{75}$$

We can insert the expressions for t-dot and φ-dot into this and get:

$$-\frac{E^2}{\left(1-\frac{r_g}{r}\right)}+\frac{1}{1-\frac{r_g}{r}}\dot{r}^2+\frac{L^2}{r^2}=0 \; . \tag{76}$$

Demonstration of V_{eff}

Now we rearrange this a little bit (and also divide everything by 2):

$$\frac{E^2}{2} = \frac{1}{2}\dot{r}^2 + \left(1 - \frac{r_g}{r}\right)\frac{L^2}{2r^2} \ . \tag{77}$$

Here we essentially have a formula for the total energy (at least qualitatively) in the form of "kinetic energy+potential energy" as follows:

$$E_{tot} \sim E_{kin}\left(\dot{r}^2\right) + V_{\text{eff}}\left(r\right) \ . \tag{78}$$

Notice the similarity of this with the usual total energy you see in basic Newtonian mechanics with the kinetic energy being a function of "velocity squared" and the potential being a function of position.

We therefore define (this is really only a definition, however, it turns out to be an extremely useful definition) the effective potential as:

$$\boxed{V_{\text{eff}}\left(r\right) = \left(1 - \frac{r_g}{r}\right)\frac{L^2}{2r^2} = \frac{L^2}{2r^2} - \frac{L^2 r_g}{2r^3} = \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3} \ ,} \tag{79}$$

where M is the mass of the black hole.