Exam — Session 1

Duration: 2 h

Documents, cell phones, computers, tablets, pocket calculators, etc., are not allowed

The text contains 4 pages in total

1 Ferromagnetism and antiferromagnetism

Let us consider a d-dimensional Ising model, consisting of $N \gg 1$ Ising spins $s_i = \pm 1$ at the temperature T, located at the sites i of a hypercubic lattice and subject to a magnetic field h (in energy units). We denote $\beta = 1/k_{\mathrm{B}}T$, with k_{B} the Boltzmann constant. In what follows, we only consider interactions between nearest neighbors. The Hamiltonian of the system is written as

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_{i=1}^{N} s_i, \tag{1.1}$$

where $\langle i, j \rangle$ denotes a summation over nearest neighbors i and j.

1.1 Ferromagnetism and mean-field approximation

In this first part of the problem, the coupling constant J is positive and we denote it $J = J_{\mathrm{F}}$, with $J_{\mathrm{F}} > 0$.

(a) Justify that within the mean field approximation,

$$s_i s_j \simeq (s_i + s_j)m - m^2 \quad \text{for} \quad i \neq j,$$

where $m = \langle s_i \rangle$ is the average magnetization per site.

(b) Deduce that within the above-mentioned approximation, the Hamiltonian (1.1) takes the form

$$H \simeq -(h + zJ_{\mathrm{F}}m) \sum_{i=1}^{N} s_i + \frac{1}{2} N z J_{\mathrm{F}} m^2, \tag{1.2}$$

with z the number of nearest neighbors of a given lattice site i.

(c) What is the physical meaning of the term $h + zJ_{\mathrm{F}}m$ in the mean-field Hamiltonian (1.2)?

(d) Calculate the canonical partition function Z and the free energy F of the system within the mean-field approximation.

(e) Show that the average magnetization m per site is the solution of a self-consistent equation that you will explicitly determine. (Do not discuss the general possible solutions.)

(f) Let us consider for this question that $h = 0$. Show that there exists a phase transition (paramagnetic-ferromagnetic) for a critical temperature T_{c}. Determine T_{c} as a function of the different parameters of the problem. What does the mean-field approximation predict for the case $d = 1$? Compare to your knowledge of the exact solution of the one-dimensional Ising model.

1.2 Antiferromagnetism

In this second part of the problem, the coupling constant is negative, and we denote it $J = -J_{\mathrm{AF}}$ with $J_{\mathrm{AF}} > 0$.

1.2.1 General results

(a) Describe the effect of the first term of the Hamiltonian (1.1) on the spin orientations.

(b) Let us consider for this question that $T = 0$ and $h = 0$. Justify that the system splits into two sublattices A and B, such that the spins take the value $+1$ or -1 depending on the sublattice to which they belong. These states are called *Néel states*. How many Néel states are there? Give the expressions of the magnetization and the average energy of the Néel states.

(c) Still at $T = 0$, what is the qualitative effect of a positive uniform magnetic field h? By comparing the energy of a Néel state subject to a finite magnetic field and that of a ferromagnetic state (where all the spins are orientated in the same direction), deduce the critical value $h_c\,(T = 0)$ for which it is possible for the system to go from the antiferromagnetic phase to the ferromagnetic one.

1.2.2 Mean-field approximation

We call $m_A = \langle s_i \rangle$ $(i \in A)$ the average magnetization of the spins belonging to the sublattice A and $m_B = \langle s_j \rangle$ $(j \in B)$ the average magnetization of the spins belonging to the sublattice B.

(a) Justify that

$$s_i s_j \simeq s_i m_B + s_j m_A - m_A m_B \qquad (i \in A, j \in B)$$

in the mean-field approximation.

(b) Deduce from the preceding question that one can approximate the Hamiltonian (1.1) by

$$H \simeq -\left(h - zJ_{\mathrm{AF}} m_B\right) \sum_{i \in A} s_i - \left(h - zJ_{\mathrm{AF}} m_A\right) \sum_{j \in B} s_j - \frac{1}{2} N z J_{\mathrm{AF}} m_A m_B.$$

(c) Using your answers to questions 1.1(c) and 1.1(e), argue that m_A and m_B verify the following system of self-consistent equations:

$$\begin{aligned} m_A &= \tanh\left(\beta\left[h - \lambda m_B\right]\right), \\ m_B &= \tanh\left(\beta\left[h - \lambda m_A\right]\right). \end{aligned}$$

What is the expression of the constant λ?

(d) Let us first consider the zero magnetic field case ($h = 0$).

(i) Assuming that $m_A = -m_B$, show that there exists a phase transition for a temperature T_{N} (called the Néel temperature) between a phase where $m_A = -m_B = 0$ and a phase where $m_A(T) = -m_B(T) = m_0(T)$. Give an expression for T_{N}. Sketch $m_A(T)$ as a function of temperature.

(ii) Sketch $m_+ = (m_A + m_B)/2$ and $m_- = (m_A - m_B)/2$ as a function of T. Which quantity is the order parameter of the antiferromagnetic-paramagnetic phase transition? What would you find if you would measure the average magnetization of the sample?

(e) We now seek to characterize the effect of the magnetic field on m_A and m_B by calculating the magnetic susceptibility of the crystal defined by

$$\chi = \left.\frac{\partial m_+}{\partial h}\right|_{h=0}.$$

(i) We first consider that $T > T_{\mathrm{N}}$ and we assume that the magnetic field is weak (with respect to what?). Linearize the self-consistent equations and show that

$$\chi\left(T\right) = \frac{C}{k_{\mathrm{B}} T + k_{\mathrm{B}} T_{\mathrm{N}}}, \tag{1.3}$$

where C is a dimensionless constant that you will determine.

(ii) We now move to the case $T < T_{\rm N}$. We assume that the magnetic field h is weak and we write the magnetizations on the sites A and B as $m_A = m_0 + \Delta m_A$ and $m_B = -m_0 + \Delta m_B$, with $\Delta m_A \ll m_0$ and $\Delta m_B \ll m_0$. By performing a Taylor expansion of the self-consistent equations, show that the susceptibility takes the form[1]

$$\chi(T) = \frac{1}{k_{\rm B}T\cosh^2\left(\frac{T_{\rm N}}{T}m_0(T)\right) + k_{\rm B}T_{\rm N}}.$$

Show that for $T > T_{\rm N}$ one finds the previous result of Eq. (1.3). How does χ behave at low temperature? Sketch the graph of $\chi(T)$ and compare it to that of a ferromagnet.

2 Landau diamagnetism of a two-dimensional electron gas

The magnetic properties of a noninteracting electron gas are controlled by two phenomena: the Pauli *paramagnetism* due to the alignment of the electronic magnetic moments with the applied magnetic field, and the Landau *diamagnetism* induced by the orbital motion of the electronic charges. In this problem we aim at describing the second of these phenomena, using the one-electron Hamiltonian (in cgs units)

$$H = \frac{1}{2m}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2, \tag{2.1}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential, $-e$ the electronic charge ($e > 0$), and c the speed of light in vacuum. In Eq. (2.1), m is the (effective) mass of the charge carriers (i.e., the electrons).

In what follows, we consider a homogeneous magnetic field B parallel to the z axis ($B = \partial_x A_y - \partial_y A_x$ = constant), and we assume that electrons are confined to a two-dimensional rectangular surface with area $\mathcal{A} = L_x L_y$, where L_x and L_y are the lateral dimensions of the electron gas in the x and y directions, respectively.

2.1 General results for noninteracting fermions

(a) Carefully demonstrate that the grand-canonical partition function for noninteracting fermions is given by

$$\Xi = \prod_\lambda \left[1 + \mathrm{e}^{-\beta(\epsilon_\lambda - \mu)}\right],$$

where the product runs over quantum states λ with energy ϵ_λ. Here, $\beta = 1/k_{\rm B}T$ with T the temperature and μ is the chemical potential.

(b) Deduce from the previous result that the general expression of the grand potential for noninteracting fermionic particles is given by

$$\Omega = -k_{\rm B}T\sum_\lambda \ln\left(1 + \mathrm{e}^{-\beta(\epsilon_\lambda - \mu)}\right). \tag{2.2}$$

2.2 Landau susceptibility

The energy spectrum of the Hamiltonian (2.1) corresponds to the one of a harmonic oscillator with (cyclotron) frequency $\omega_{\rm c} = eB/mc$ (we assume $B > 0$ from now on):

$$\epsilon_n = \hbar\omega_{\rm c}\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{N},$$

defining so-called *Landau levels*. Each Landau level is highly degenerate, with degeneracy factor (including the spin degeneracy)

$$g_n = \rho_0\hbar\omega_{\rm c},$$

where $\rho_0 = m\mathcal{A}/\pi\hbar^2$ is the density of states of the two-dimensional electron gas at zero magnetic field.[2]

[1] We recall that $\tanh(a + x) \simeq \tanh a + x/\cosh^2 a$ for $x \ll 1$.
[2] Note that the degeneracy factor is in fact independent on n.

(a) Give an expression of the grand-potential (2.2) in terms of a summation over Landau levels n and as a function of ρ_0.

(b) The Euler–MacLaurin formula allows one to approximate a discrete summation by the following expression:

$$a\sum_{n=0}^{\infty} f\big(a(n+1/2)\big) \underset{(a\ll 1)}{=} \int_0^{\infty} \mathrm{d}x\, f(x) + \frac{a^2}{24} f'(0) + \mathcal{O}\left(a^3\right),$$

where $f(x)$ is a function that decreases sufficiently fast when $x \to \infty$, where $f'(x)$ is its derivative with respect to x, and where a is some dimensionless parameter. Use the above formula to show, in the limits $\beta\hbar\omega_{\mathrm{c}} \ll 1$ and $\beta\mu \gg 1$, that

$$\Omega(B) \simeq \Omega(B=0) + \frac{\rho_0}{24}\left(\hbar\omega_{\mathrm{c}}\right)^2,$$

where the expression for $\Omega(B=0)$ involves an integral not to be calculated.

(c) Let us define the Landau susceptibility as

$$\chi_{\mathrm{L}} = -\frac{1}{\mathcal{A}} \lim_{B\to 0} \frac{\partial^2 \Omega}{\partial B^2}.$$

Show that

$$\chi_{\mathrm{L}} = -\frac{e^2}{12\pi m c^2}.$$