Exam — Session 2

Duration: 2 h

Documents, cell phones, computers, tablets, pocket calculators, etc., are not allowed

The text contains 4 pages in total

1 Landau diamagnetism of a two-dimensional electron gas

The magnetic properties of a noninteracting electron gas are controlled by two phenomena: the Pauli *paramagnetism* due to the alignment of the electronic magnetic moments with the applied magnetic field, and the Landau *diamagnetism* induced by the orbital motion of the electronic charges. In this problem we aim at describing the second of these phenomena, using the one-electron Hamiltonian (in cgs units)

$$H = \frac{1}{2m}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2, \tag{1.1}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential, $-e$ the electronic charge ($e > 0$), and c the speed of light in vacuum. In Eq. (1.1), m is the (effective) mass of the charge carriers (i.e., the electrons).

In what follows, we consider a homogeneous magnetic field B parallel to the z axis ($B = \partial_x A_y - \partial_y A_x = \text{constant}$), and we assume that electrons are confined to a two-dimensional rectangular surface with area $\mathcal{A} = L_x L_y$, where L_x and L_y are the lateral dimensions of the electron gas in the x and y directions, respectively.

1.1 General results for noninteracting fermions

(a) Carefully demonstrate that the grand-canonical partition function for noninteracting fermions is given by

$$\Xi = \prod_\lambda \left[1 + \mathrm{e}^{-\beta(\epsilon_\lambda - \mu)}\right],$$

where the product runs over quantum states λ with energy ϵ_λ. Here, $\beta = 1/k_\mathrm{B}T$ with T the temperature and k_B the Boltzmann constant, and μ is the chemical potential.

(b) Deduce from the previous result that the general expression of the grand potential for noninteracting fermionic particles is given by

$$\Omega = -k_\mathrm{B}T \sum_\lambda \ln\left(1 + \mathrm{e}^{-\beta(\epsilon_\lambda - \mu)}\right). \tag{1.2}$$

1.2 Landau susceptibility

The energy spectrum of the Hamiltonian (1.1) corresponds to the one of a harmonic oscillator with (cyclotron) frequency $\omega_\mathrm{c} = eB/mc$ (we assume $B > 0$ from now on):

$$\epsilon_n = \hbar\omega_\mathrm{c}\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{N},$$

defining so-called *Landau levels*. Each Landau level is highly degenerate, with degeneracy factor (including the spin degeneracy)

$$g_n = \rho_0 \hbar\omega_\mathrm{c},$$

where $\rho_0 = m\mathcal{A}/\pi\hbar^2$ is the density of states of the two-dimensional electron gas at zero magnetic field.[1]

[1]Note that the degeneracy factor is in fact independent on n.

(a) Give an expression of the grand-potential (1.2) in terms of a summation over Landau levels n and as a function of ρ_0.

(b) The Euler–MacLaurin formula allows one to approximate a discrete summation by the following expression:

$$a\sum_{n=0}^{\infty} f\big(a(n+1/2)\big) \underset{(a\ll 1)}{=} \int_0^{\infty} \mathrm{d}x\ f(x) + \frac{a^2}{24} f'(0) + \mathcal{O}\left(a^3\right),$$

where $f(x)$ is a function that decreases sufficiently fast when $x \to \infty$, where $f'(x)$ is its derivative with respect to x, and where a is some dimensionless parameter. Use the above formula to show, in the limits $\beta\hbar\omega_{\mathrm{c}} \ll 1$ and $\beta\mu \gg 1$, that

$$\Omega(B) \simeq \Omega(B=0) + \frac{\rho_0}{24}\left(\hbar\omega_{\mathrm{c}}\right)^2,$$

where the expression for $\Omega(B=0)$ involves an integral not to be calculated.

(c) Let us define the Landau susceptibility as

$$\chi_{\mathrm{L}} = -\frac{1}{\mathcal{A}} \lim_{B\to 0} \frac{\partial^2 \Omega}{\partial B^2}.$$

Show that

$$\chi_{\mathrm{L}} = -\frac{e^2}{12\pi m c^2}.$$

2 The Blume–Capel model

The Blume–Capel model describes a magnetic material with some nonmagnetic vacancies. Let us consider a lattice [we denote by $N(\gg 1)$ the number of lattice sites and by z the number of nearest neighbors] of spins S_i that can take the values -1, 0, and $+1$. A spin 0 corresponds to a vacancy (nonmagnetic impurity or empty site) and spins $+1$ or -1 correspond to the two different orientations of the magnetic species. We assume that the Hamiltonian of the system in presence of an homogeneous magnetic field h is given by

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_i S_j + \Delta \sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i, \tag{2.1}$$

where $J > 0$ is the exchange interaction and where Δ is a constant that can be either negative or positive. In the Hamiltonian above, $\langle i,j\rangle$ denotes a summation over nearest neighbors.

2.1 General discussion

(a) Justify that $-\Delta$ is the energy of creation of a vacancy. In which case ($\Delta > 0$ or $\Delta < 0$) is it favorable to create a vacancy?

(b) At $T = 0$ and $h = 0$, calculate the energy of the system in the three different states $\langle S_i\rangle = +1$, $\langle S_i\rangle = -1$, and $\langle S_i\rangle = 0$. Which state is selected at $T = 0$?

(c) Which limit of Δ corresponds to the usual two-state Ising model? How would you call the $\Delta = 0$ model?

2.2 Mean-field approximation

We now aim at performing a mean-field approximation (MFA). We write $S_i = m + \delta S_i$, where $m = \langle S_i\rangle$ is the average magnetization and δS_i the fluctuations of the spin S_i around m.

(a) Define the spin-spin correlation function C_{ij}. What is the value of C_{ij} in the MFA?

(b) Show that within the MFA, it is possible to write the Hamiltonian (2.1) as

$$\mathcal{H} \simeq \frac{1}{2}NzJm^2 - (h + zJm)\sum_{i=1}^{N} S_i + \Delta \sum_{i=1}^{N} S_i^2.$$

(c) Calculate the free energy F within the MFA.

(d) Demonstrate that the average value $m = \langle S_i \rangle$ is given by the expression

$$m = -\frac{1}{N}\frac{\partial F}{\partial h}.$$

Deduce that, within the MFA, the magnetization obeys the self-consistent equation (SCE)

$$m = \frac{2\sinh\left(\beta[h + zJm]\right)}{\exp\left(\beta\Delta\right) + 2\cosh\left(\beta[h + zJm]\right)}.$$

From now on, we consider the case of vanishing magnetic field, $h = 0$.

(e) In the case $\Delta \to -\infty$, discuss the solutions of the SCE.

(f) In the general case, show that $m = 0$ is a solution of the SCE.

(g) We now aim at discussing graphically the solutions of the SCE. We define $t = k_{\mathrm{B}}T/zJ$ and $\delta = \Delta/zJ$.

(i) Express the SCE in term of the function

$$f(m) = \frac{2\sinh\left(m/t\right)}{\exp\left(\delta/t\right) + 2\cosh\left(m/t\right)}.$$

(ii) What is the value of $f(0)$?

(iii) What are the limits of $f(m)$ when $m \to \pm\infty$?

(iv) Calculate

$$\left.\frac{\mathrm{d}f}{\mathrm{d}m}\right|_{m=0}$$

and discuss graphically the number of solutions to the SCE. Show that there is a critical reduced temperature t_{c} defined by the equation

$$t_{\mathrm{c}} = \frac{2}{2 + \exp\left(\delta/t_{\mathrm{c}}\right)}.$$

(v) On next page in figure 1 (colored lines) is plotted the function

$$g(t, \delta) = \frac{2}{2 + \exp\left(\delta/t\right)} \qquad (2.2)$$

as a function of t for different values δ_i of δ. Which δ_i's are positive and which of them are negative? Sort by ascending order the δ_i's.

(vi) Plot the curve $g(t, \delta)$ for the value of δ corresponding to the Ising model and give the corresponding t_{c}.

(vii) Using your previous discussion and question 2.1(b), sketch the general behavior of t_{c} as a function of δ.

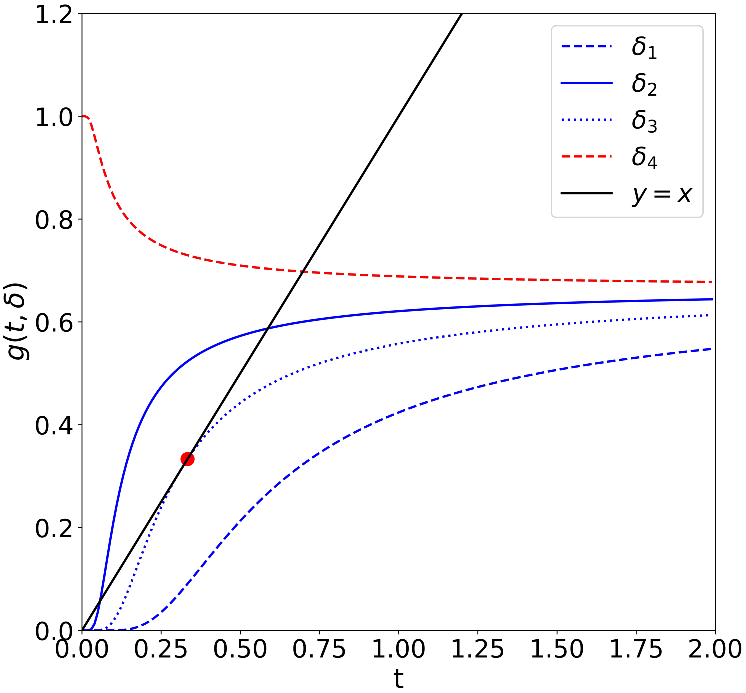

Figure 1: Colored lines: Plot of $g(t, \delta)$ as defined in Eq. (2.2) as a function of t for different values δ_i of δ. Black solid line: t.