Problem Set 1
The free electron gas

1 Particle in a box

Let us consider a free, nonrelativistic quantum particle with mass m and spin s confined in a cubic box with sides of length L.

(a) Calculate the particle' wavefunctions and corresponding quantized energies for the cases of (i) hard wall and (ii) periodic boundary conditions.

(b) Give (without any justification) the spin degeneracy factor η_s of each orbital eigenstate. How much is η_s for electrons, that are spin $s = 1/2$ elementary particles?

(c) Considering from now on electrons, and using periodic boundary conditions, estimate the number $\mathcal{N}(E)$ of quantum states in the box that have an energy lower than E, in the regime where the box is sufficiently large such that $L \gg 2\pi\hbar/\sqrt{2mE}$. Show that in this limit, $\mathcal{N}(E)$ is proportional to the volume $V = L^3$ of the box, and that it does not depend on the choice of the boundary conditions.

(d) Use the result of question (c) to calculate the density of states per unit volume $g(E)$ (*i.e.*, the number of quantum states per unit energy and per unit volume).

(e) Determine the density of states per unit surface and length, respectively, for free electrons in two and one dimensions.

2 Three-dimensional electron gas

(a) Use the results of Exercise 1 to express the Fermi energy E_F, the Fermi wavevector k_F, and the Fermi velocity v_F of a three-dimensional electron gas as a function of the electron density n.

(b) Show that the total kinetic energy of a three-dimensional electron gas containing N free electrons at zero temperature is given by $E_\mathrm{kin} = \frac{3}{5}NE_\mathrm{F}$.

(c) The spatial density of atoms in copper is $8.45 \times 10^{28}\,\mathrm{m}^{-3}$. Estimate the values of the quantities in question (a) for the conduction band of copper, assuming a free electron gas and one conduction band electron per atom. Use the bare electron mass $m_\mathrm{e} = 9.11\times10^{-31}\,\mathrm{kg}$, and Planck's constant $\hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$ ($1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$).