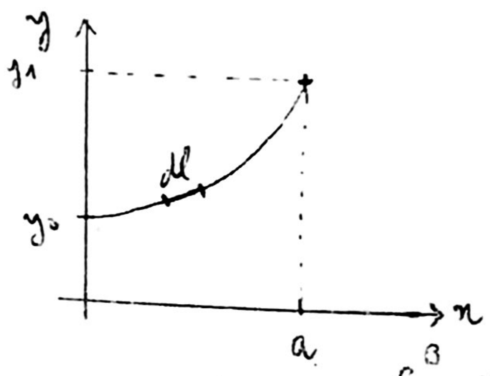
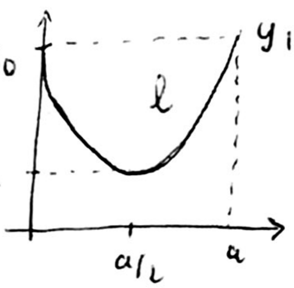
b. Retour sur la chaînette (avec contrainte)



cas particulier



y_{min} dépend de la longueur ℓ

- Energie potentielle : $U_p = \int \rho g \, dl \, y = \int_{x_A}^{x_B} y \rho g \, dx \sqrt{1+\dot{y}^2}$
- contrainte : $\ell = \int_{x_A}^{x_B} dl = \int_{x_A}^{x_B} \sqrt{1+\dot{y}^2} \, dx$

On définit

$$h = f - \lambda g$$
$$= \rho g y \sqrt{1+\dot{y}^2} - \lambda \sqrt{1+\dot{y}^2} = (\rho g y - \lambda)\sqrt{1+\dot{y}^2}$$

h ne dépend pas explicitement de x : on utilise Beltrami

$$h - \dot{y}\frac{\partial h}{\partial \dot{y}} = k \iff (\rho g y - \lambda)\sqrt{1+\dot{y}^2} - \dot{y} \times (\rho g y - \lambda)\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = k$$
$$\iff (\rho g y - \lambda)(1+\dot{y}^2) - \dot{y}^2(\rho g y - \lambda) = k\sqrt{1+\dot{y}^2}$$
$$\iff (\rho g y - \lambda) = k\sqrt{1+\dot{y}^2}$$
$$\iff (\rho g y - \lambda)^2 = k^2 + k^2\dot{y}^2$$
$$\iff \dot{y}^2 = \left(\frac{dy}{dx}\right)^2 = \frac{(\rho g y - \lambda)^2 - k^2}{k^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{(\rho g y - \lambda)^2 - k^2}}{k}$$

On intègre :

$$\int dx = \int \frac{k \, dy}{\sqrt{(\rho g y - \lambda)^2 - k^2}} = \int \frac{k \cancel{k} \sinh\theta \, d\theta}{\cancel{\rho g k}\sqrt{\cosh^2\theta - 1}} = \int d\theta$$

On pose

$(\rho g y - \lambda) = k \cosh\theta$ et $\cosh^2\theta - \sinh^2\theta = 1$

$\rho g \, dy = k \sinh\theta \, d\theta$; $\cosh^2\theta - 1 = \sinh^2\theta$

→

$x = \frac{k}{\rho g}\theta + c$ et $(\rho g y - \lambda) = k\cosh\theta$

$\frac{\rho g}{k}(x - x_c) = \operatorname{argcosh}\left(\frac{\rho g y - \lambda}{k}\right)$ $\quad \cdot\ \theta = \operatorname{argcosh}\left(\frac{\rho g y - \lambda}{k}\right)$

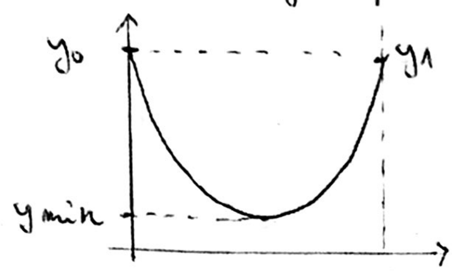
$\cdot\ c = x_c$

$$\Leftrightarrow (\rho g y - \lambda) = k\cosh\left(\frac{\rho g(x - x_c)}{k}\right)$$

$$\boxed{y(x) = \frac{k}{\rho g}\cosh\left(\frac{\rho g}{k}(x - x_c)\right) + \frac{\lambda}{\rho g}}$$

ajout de la contrainte.

Regardons le cas symétrique



$$\begin{cases} y_0 = y(0) = \frac{k}{\rho g}\cosh\left(\frac{-x_c\rho g}{k}\right) + \frac{\lambda}{\rho g} \\ \quad = \frac{k}{\rho g}\cosh\left(\frac{x_c\rho g}{k}\right) + \frac{\lambda}{\rho g} \\ y_1 = y(a) = \frac{k}{\rho g}\cosh\left(\frac{(a - x_c)\rho g}{k}\right) + \frac{\lambda}{\rho g} \end{cases}$$

Ces c. limites implique

$$\left.\begin{aligned} x_c &= a - x_c \\ 2x_c &= a \end{aligned}\right\} \quad x_c = a/2$$

$$\boxed{y(x) = \frac{k}{\rho g}\cosh\left(\frac{\rho g}{k}(x - a/2)\right) + \frac{\lambda}{\rho g}}$$

En $y = a/2$ $\quad y(a/2) = y_{min} = \frac{k}{\rho g}\cosh(0) + \frac{\lambda}{\rho g} = \frac{k}{\rho g} + \frac{\lambda}{\rho g}$

$$\boxed{y_{min} = \frac{k}{\rho g} + \frac{\lambda}{\rho g}}$$

<u>Contrainte</u>: La longueur est $l = \int_{x=0}^{x=a} dl = \int_0^a \sqrt{1+\dot{y}^2}\,dx$

et $\dot{y} = \frac{dy}{dx} = \frac{k}{\rho g}\frac{d}{dx}\left(\cosh\left(\frac{\rho g}{k}(x - a/2)\right)\right) = \frac{k}{\rho g}\times\frac{\rho g}{k}\sinh\left(\frac{\rho g}{k}(x - a/2)\right)$

$$l = \int_0^a \left(1 + \sinh^2\left(\frac{\rho g}{k}(x - a/2)\right)\right)^{1/2} dx = \int_0^a \cosh\left(\frac{\rho g}{k}(x - a/2)\right) dx$$

$ch^2 - sh^2 = 1 \Leftrightarrow ch^2 = sh^2 + 1$ $\quad = \left[\frac{k}{\rho g}\sinh\left(\frac{\rho g}{k}(x - a/2)\right)\right]_0^a$ →

$$\ell = \frac{k}{\rho g}\left(\sinh\left(\frac{\rho g a}{k\,2}\right) - \sinh\left(\rho g \frac{-a}{2}\right)\right)$$

$$\boxed{\ell = \frac{2k}{\rho g}\sinh\left(\frac{\rho g a}{2k}\right)}$$ longueur de la corde.

$$\begin{cases} y_1 = \dfrac{k}{\rho g}\cosh\left(\dfrac{a\rho g}{2k}\right) + \dfrac{\lambda}{\rho g} & (1) \\ y_{min} = \dfrac{k}{\rho g} + \dfrac{\lambda}{\rho g} & (2) \\ \ell = \dfrac{2k}{\rho g}\sinh\left(\dfrac{a\rho g}{2k}\right) & (3) \end{cases}$$

inconnues λ, k, y_{min}.

$$\begin{cases} (3) \text{ donne } k \\ (1) \text{ et } k \text{ donne } \lambda \\ (2),\ k \text{ et } \lambda \text{ donnent } y_{min}. \end{cases}$$