Problem Set 8
Orbital magnetism

1 Landau levels

Let us consider a two-dimensional gas of free, noninteracting electrons confined to the x-y plane in the presence of an external magnetic field perpendicular to the gas, *i.e.*, $\mathbf{B} = B\,\hat{z}$. In what follows, we do not take into account the electronic spin, and consider electrons as particles with charge $-e < 0$ and (effective) mass m_*.

(a) Write the classical equations of motion for an electron in the x-y plane and show that the solutions are circular orbits. Determine the angular velocity ω_c (*i.e.*, the cyclotron frequency) and the cyclotron radius R_c of the classical motion in these orbits.

(b) Let us use Landau's gauge $\mathbf{A} = A\,\hat{x}$ for the vector potential. Determine A.

(c) Write the time-independent Schrödinger equation for an electron in the x-y plane.

(d) Use the product ansatz $\psi_{n,k_x}(x, y) = \varphi_n^{(k_x)}(y)\,\mathrm{e}^{\mathrm{i}k_x x}$ to relate the Schrödinger equation to the (quantum) harmonic oscillator problem. Express $\varphi_n^{(k_x)}(y)$ in terms of shifted harmonic oscillator eigenfunctions. Argue that the corresponding energy levels are given by $\varepsilon_n = \hbar\omega_\mathrm{c}(n + 1/2)$ with $n \in \mathbb{N}$ and independent of k_x. Such levels are called *Landau levels*. What is the group velocity corresponding to the planewave component in the x direction?

(e) Consider a rectangular system of size $\mathcal{A} = L_x \times L_y$ with periodic boundary conditions. Discuss the allowed values for k_x and show that the degeneracy of each Landau level is given by $\mathcal{N}_\mathrm{LL} = m_*\omega_\mathrm{c}\mathcal{A}/2\pi\hbar$. Give an estimation of $\mathcal{N}_\mathrm{LL}$ for a sample with area $\mathcal{A} = 1\,\mathrm{cm}^2$ in a magnetic field $B = 0.1\,\mathrm{T}$. Give a physical interpretation of the degeneracy.

(f) Calculate the magnetic flux through the system and express it in units of the flux quantum $\phi_0 = h/e$. Comment the result in the context of the degeneracy of the Landau levels.

(g) Compare the average density of states (per unit surface) to the density of states in the absence of a magnetic field.

2 Landau diamagnetism

We now consider electrons in three dimensions in an external magnetic field $\mathbf{B} = B\hat{z}$ oriented along the z axis, and continue to ignore the spin degree of freedom.

(a) Extend the product ansatz of Exercise 1 towards the three-dimensional case in a large rectangular cuboid of size $\mathcal{V} = L_x \times L_y \times L_z$ and show that, when $B \neq 0$, the energy of an electron depends on the quantum number n and on the wavenumber k_z in the z direction as

$$\varepsilon_{n,k_z} = \hbar\omega_\mathrm{c}\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m_*}.$$

(b) Let us treat the problem within the grand-canonical ensemble. Write the grand partition function Ξ as a sum over many-body microstates l characterized by the total particle number N_l and energy E_l. Show that in the case of noninteracting fermions one can write the grand-potential $\Omega = -k_\mathrm{B}T\ln\Xi$ in the form

$$\Omega = -k_\mathrm{B}T\sum_\lambda \ln\left(1 + \mathrm{e}^{-\beta(\epsilon_\lambda - \mu)}\right)$$

as a sum over one-body states λ.

(c) Write the grand-potential as a sum over the Landau level index n and k_z. Replace the sum over k_z by an integral and show that Ω can be written as

$$\Omega = \hbar\omega_\mathrm{c} \sum_{n=0}^{\infty} f(\mu - \hbar\omega_\mathrm{c}[n+1/2]). \tag{1}$$

Determine the function $f(E)$.

(d) Apply the variant of the Euler-MacLaurin formula

$$\sum_{n=0}^{\infty} F(n+1/2) \simeq \int_0^\infty \mathrm{d}x\, F(x) + \frac{1}{24} F'(0)$$

to the sum over n in Eq. (1). In which limit is this formula a good approximation? Show that the result can be expressed as

$$\Omega \simeq \Omega_0(\mu) - \frac{(\hbar\omega_\mathrm{c})^2}{24} \frac{\partial^2 \Omega_0(\mu)}{\partial \mu^2},$$

with Ω_0 the grand potential in the absence of a magnetic field.

(e) Use the result for Ω to relate the magnetic susceptibility per volume

$$\chi = -\frac{\mu_0}{V} \frac{\partial^2 \Omega}{\partial B^2}$$

to the density of states (including the spin degeneracy) per unit volume at the chemical potential $g(\mu)$. Show that the resulting Landau susceptibility corresponds to a diamagnetic behavior and is given by

$$\chi_\mathrm{L} = -\frac{1}{3} \left(\frac{m_\mathrm{e}}{m_*} \right)^2 \chi_\mathrm{P},$$

with m_e the bare electron mass and where $\chi_\mathrm{P} = \mu_0 \mu_\mathrm{B}^2 g(\mu)$ is the paramagnetic Pauli susceptibility (which is due to the spin of the electron), with $\mu_\mathrm{B} = e\hbar/2m_\mathrm{e}$ Bohr's magneton.

(f) Discuss your result for the total magnetic susceptibility (including orbital and spin degrees of freedom) of the free electron gas $\chi = \chi_\mathrm{L} + \chi_\mathrm{P}$.