Exercises

Disclaimer: The following exercises are adapted and/or reproduced from the excellent textbook by D.J. Griffiths, *Introduction to Quantum Mechanics* (Cambridge University Press). Several copies of the book can be found at the University library.

— Chapter 1 —

Exercise 1.1: Normalization

Consider the wave function

$$\psi(x,t) = A\,\mathrm{e}^{-\lambda|x|}\,\mathrm{e}^{-\mathrm{i}\omega t},$$

where A, λ, and ω are positive real constants. (We will see in Chapter 2 what potential V actually produces such a wave function.)

(a) Normalize ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation σ of x. Sketch the graph of $|\psi|^2$, as a function of x, and mark the points $\langle x\rangle + \sigma$ and $\langle x\rangle - \sigma$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Exercise 1.2: Momentum

Ehrenfest's theorem states that expectation values obey classical laws. Prove in particular that

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \left\langle -\frac{\partial V}{\partial x}\right\rangle.$$

Exercise 1.3: The uncertainty principle

A particle of mass m is in the state

$$\psi(x,t) = A\,\mathrm{e}^{-a(mx^2/\hbar + \mathrm{i}t)},$$

where A and a are positive real constants.

(a) Find A. *Hint:* $\int_{-\infty}^{+\infty}\mathrm{d}u\,\mathrm{e}^{-u^2} = \sqrt{\pi}$.

(b) For what potential energy function $V(x)$ does ψ satisfy the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2. *Hint:* $\int_{-\infty}^{+\infty}\mathrm{d}u\,u^2\,\mathrm{e}^{-u^2} = \sqrt{\pi}/2$.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

— CHAPTER 2 —

Exercise 2.1: Stationary states

Prove the following four theorems:

(a) For normalizable solutions, the separation constant E must be real. *Hint:* Write E in $\psi(x,t) = \varphi(x)\,\mathrm{e}^{-\mathrm{i}Et/\hbar}$ as $E_0 + \mathrm{i}\Gamma$ (with E_0 and Γ real), and show that if

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ |\psi(x,t)|^2 = 1$$

is to be hold for all t, Γ must be zero.

(b) The time-independent wave function $\varphi(x)$ can always be taken to be *real* [unlike $\psi(x,t)$, which is necessarily complex.] This does not mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to φ's that are real. *Hint:* If $\varphi(x)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + V(x)\varphi(x) = E\varphi(x) \tag{1}$$

for a given E, so too does its complex conjugate, and hence also the real linear combination $\varphi + \varphi^*$ and $\mathrm{i}\,(\varphi - \varphi^*)$.

(c) If $V(x)$ is an *even function* [*i.e.*, $V(x) = V(-x)$] then $\varphi(x)$ can always be taken to be either even or odd. *Hint:* If $\varphi(x)$ satisfies Eq. (1), for a given E, so too does $\varphi(-x)$, and hence also the even and odd linear combination $\varphi(x) \pm \varphi(-x)$.

Exercise 2.2: The harmonic oscillator

Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle T\rangle = \langle p^2\rangle/2m$, for the nth stationary state of the harmonic oscillator, using the method explicited in the lecture (cf. Example 2.5). Check that the uncertainty principle is satisfied.

— CHAPTER 3 —

Exercise 3.1: Hilbert space

(a) For what range of ν is the function $f(x) = x^\nu$ in Hilbert space, on the interval $[0,1]$? Assume ν is real, but not necessarily positive.

(b) For the specific case $\nu = 1/2$, is $f(x)$ in Hilbert space? What about $xf(x)$? How about $\frac{\mathrm{d}f}{\mathrm{d}x}$?

Exercise 3.2: A few commutators

(a) Prove the following commutator identities:

$$\begin{aligned}[A+B,C] &= [A,C] + [B,C]\\ [AB,C] &= A\,[B,C] + [A,C]\,B.\end{aligned}$$

(b) Show that

$$[x^n, p] = i\hbar\, n x^{n-1}.$$

(c) Show more generally that

$$[f(x), p] = i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$$

for any function $f(x)$ (which is differentiable, of course).

(d) Show that for the simple harmonic oscillator

$$[H, a_\pm] = \pm\hbar\omega a_\pm.$$

Exercise 3.3: Generalized uncertainty principle

Prove the uncertainty principle relating the uncertainty in position to the uncertainty in energy:

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} \left| \langle p \rangle \right| .$$

Exercise 3.4: Time-evolution of expectation values

In the lecture we have learned that the time evolution of the expectation value of an operator $Q(x, p, t)$ is given by

$$\frac{\mathrm{d}\langle Q \rangle}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle . \tag{2}$$

Apply Eq. (2) to the following spectial cases: (a) $Q = 1$, (b) $Q = H$, (c) $Q = x$, and (d) $Q = p$. In each case, comment on the result.

— Chapter 4 —

Exercise 4.1: Quantum mechanics in three dimensions

(a) Work out all of the *canonical commutation relations* for components of the operators $\mathbf{r}$ and $\mathbf{p}$, and show that

$$[r_i, r_j] = 0, \qquad [p_i, p_j] = 0, \qquad [r_i, p_j] = \mathrm{i}\hbar\delta_{ij},$$

where the indices stand for x, y, or z and $r_x = x$, $r_y = y$, and $r_z = z$.

(b) Confirm Ehrenfest's theorem for three dimensions:

$$\frac{\mathrm{d}\langle \mathbf{r} \rangle}{\mathrm{d}t} = \frac{\langle \mathbf{p} \rangle}{m}, \qquad \frac{\mathrm{d}\langle \mathbf{p} \rangle}{\mathrm{d}t} = \langle -\nabla V \rangle.$$

(Each of these, of course, stands for *three* equations—one for each component.) *Hint:* use Eq. (2).

(c) Formulate Heisenberg's uncertainty principle in three dimensions. *Answer:*

$$\sigma_x \sigma_{p_x} \geqslant \frac{\hbar}{2}, \qquad \sigma_y \sigma_{p_y} \geqslant \frac{\hbar}{2}, \qquad \sigma_z \sigma_{p_z} \geqslant \frac{\hbar}{2},$$

but there is no restriction on, say, $\sigma_x \sigma_{p_y}$.