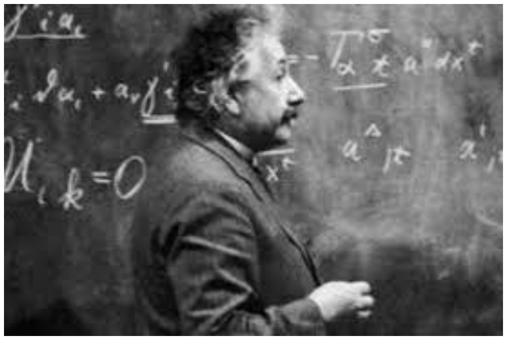
General Relativity (GR)

M1 - Physique 2023-2024



AE+GR (1907-1917)

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$$g_{\mu\nu}$$

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

II) preliminaries of physics and mathematics

- General considerations (Einstein equivalence principle…)
- Field theory
- Tensors
- Special relativity and complements

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

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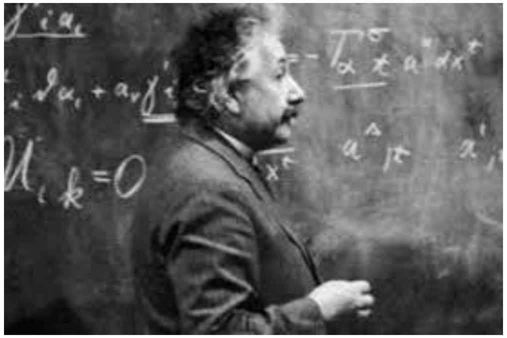

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General considerations

- General relativity is our **best current theory** describing (1) the gravitational interaction and (2) the geometric aspects of space-time. The fact that these two themes go hand in hand is characteristic of the theory's physical content.

Gravity = geometric aspects of space-time

- The theory made some astonishing predictions: **black holes**, **gravitational waves**, **expansion of the universe**, **gravitational redshift and time dilation**. They have all been verified! One of the last:

- The merger of two neutron stars was detected on August 17, 2017 in the galaxy NGC 4993, both as gravitational waves and as light. In all, in addition to **LIGO** and **Virgo**, some 70 observatories on the ground and in space took part in monitoring the event. This is the first time that gravitational waves have been detected with an electromagnetic counterpart. This detection reinforces the hypothesis that gamma-ray bursts, or at least some of them, are the result of the merger of two neutron stars. This detection verified with **an accuracy of one part in 10^{15} GR's prediction** that the two signals are moving at the same speed, thus ruling out a large number of other theories which gave different predictions.

General considerations

To this day, this theory has never been proven wrong!

- The theory's validity is limited by the fact that it does not take quantum effects into account. These are expected at scales of Planck length.

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} \quad \mathrm{m}$$

Obtained using dimensional analysis based on the three universal constants, gravitation, quantum and special relativity.

- *It represents the length scale at which a classical - non-quantum - description of gravitation ceases to be valid, and quantum mechanics must be taken into account*.
- Likely to be crucial at the center of black holes at the end of their evaporation (Hawking radiation) and at the very beginning of the universe.
- Gravity is described by a field theory like electromagnetism; but this field **also** determines what we call the properties of space-time.

General considerations

Fields

Static limit		*Complete theory*
$F_{\mathrm{em}} \propto \dfrac{e_1 e_2}{r^2}$	Coulomb law	Maxwell's electromagnetic field theory
$F_{\mathrm{g}} \propto \dfrac{m_1 m_2}{r^2}$	Newton law	General relativity (GR)

Static limit:

- compatible with the principle of Galilean relativity*;* inv. / Galilean transf.
- the interaction propagates instantaneously $c \to +\infty$

Complete theory:

- compatible with the principle of special relativity*;* inv. / Lorentz transf.
- the interaction propagates at the speed of light *c*

General considerations

→ Need for a field theory

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

	Electromagnetism	*General relativity*
Field	$A_a(\mathbf{x})$ Spin 1, Photon, Dipolar coupling Vector potential	$g_{ab}(\mathbf{x})$ Spin 2, Graviton, Quadrupolar coupling Metric tensor or gravitational potential
Equation of motion for a particle of mass *m*	$\ddot{x}^a = \frac{e}{m}F^a_b\dot{x}^b$ Lorentz force	$\ddot{x}^a = -\Gamma^a_{bc}\dot{x}^b\dot{x}^c$ Geodesic equation
Field equations	$D_aF^{ab} = 4\pi J^b$ Maxwell equation	$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$ Einstein equation

Covariant formulation!

General considerations

COVARIANCE (1)

In theoretical physics, general **covariance** (or general **invariance**) is the invariance of the form of physical laws under any differentiable coordinate transformation. The principle behind this notion is that there are no a priori coordinates in Nature, only mathematical devices used to describe it, and which should therefore play no role in the expression of the fundamental laws of physics. In other words, according to the principle underlying the notion of general covariance, physical laws do not a priori relate directly to Nature, but to an abstract differential variety. A physical law that is general covariant takes the same mathematical form in any coordinate system and is generally expressed in terms of tensor fields. The theories of electrodynamics formulated at the beginning of the 20th century are examples of this.

General considerations

COVARIANCE (2)

Albert Einstein proposed this principle for his special relativity. However, it only models coordinate systems in space-time that are linked by uniform (not accelerated) relative motion, the *inertial frame of reference*. Einstein established that the generalized principle of relativity must also apply to accelerated relative motion and used tensor calculus, a new mathematical tool at the time, to extend the general Lorentz invariance of special relativity (which applies only to inertial reference frames) to local Lorentz invariance (which applies to all reference frames). This extension enabled him to create general relativity.

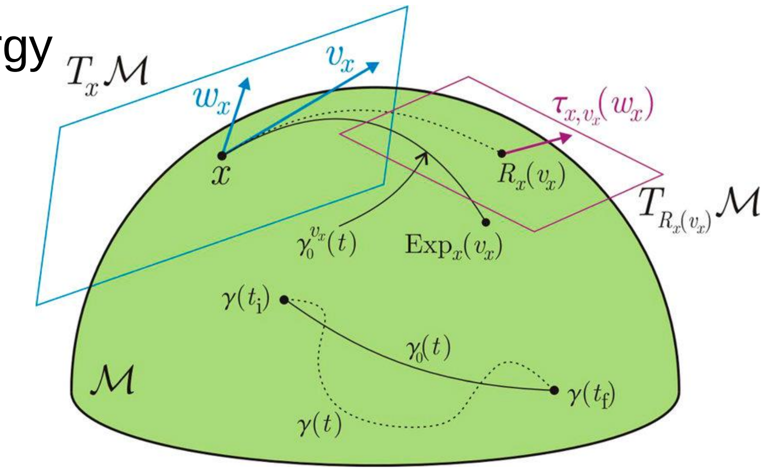
General considerations

Tools to be developed

g_{ab} ↑ metric $F^a_{\ b}$ ↑ electromagnetic R_{ab} ↑ Ricci T_{ab} ↑ Stress-energy

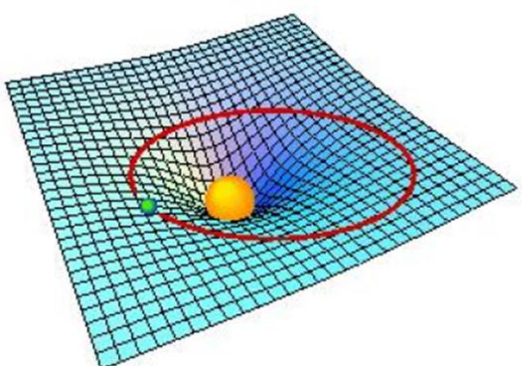
: tensors

→ **Tensor calculus**



Γ^a_{bc} → Levi-Civita **connection**

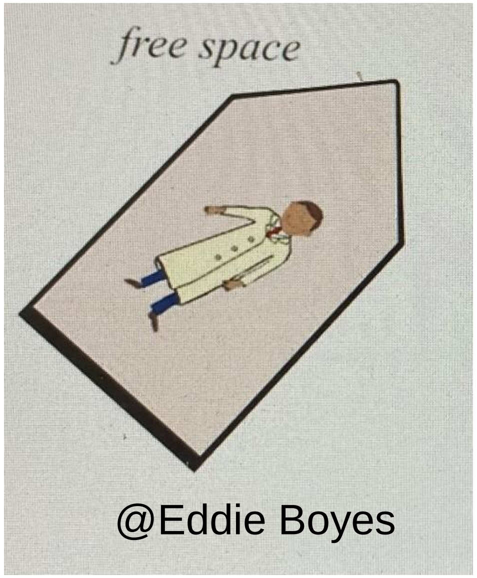
R_{ab} → **Riemannian geometry /differential geometry**



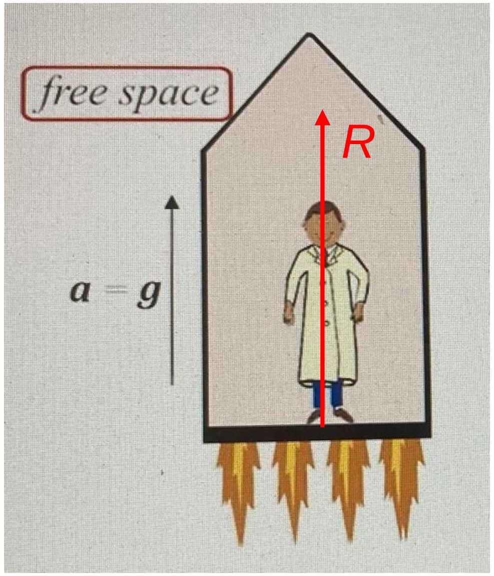
- Gravity is the result of distortions in space-time created by mass and energy

General considerations

Einstein Equivalence Principe (**EEP**)



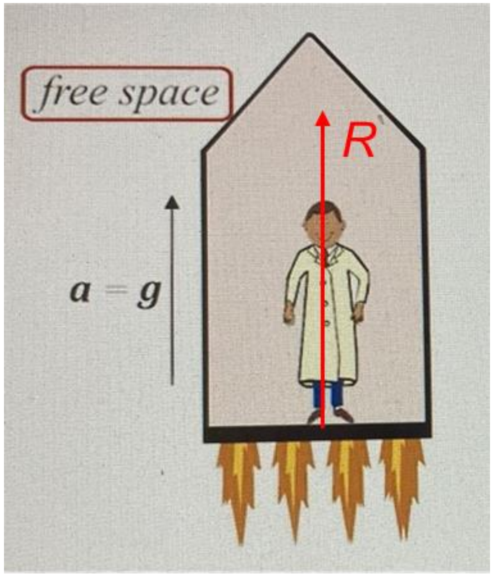
- No force acts on the spaceship
- No force acts on man
- The man can't see outside the spaceship



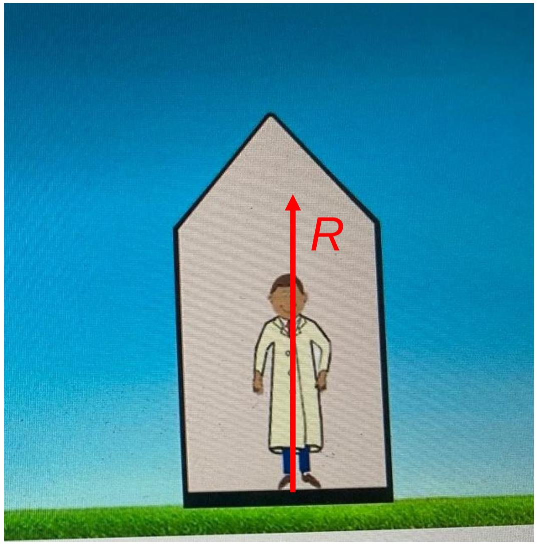
- The man is pushed towards the ground with a force equivalent in amplitude to that which would attract him to the ground on earth
- The reaction force *R* gives him the sensation of his weight

General considerations

Einstein Equivalence Principe (**EEP**)



These two situations are **indistinguishable**



earth

→ **Gravity is the same as accelaration**

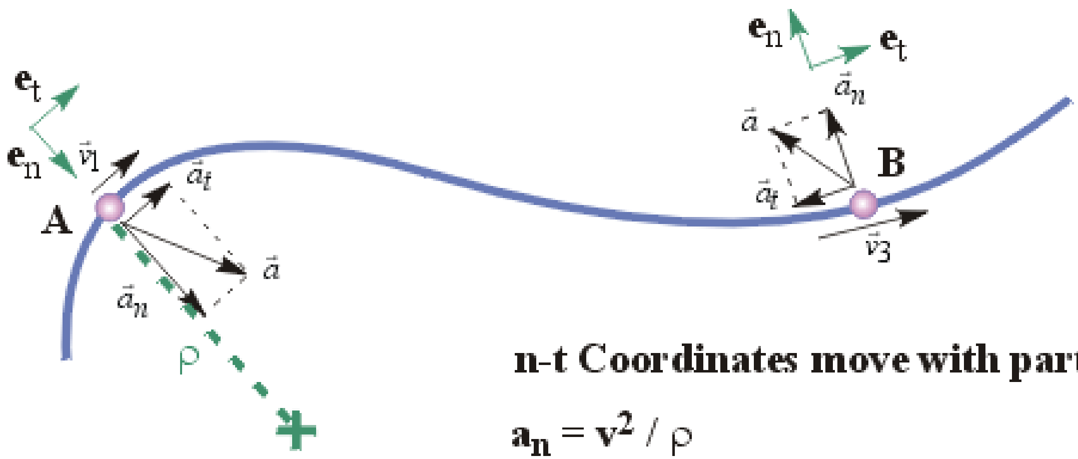
- This principle is at the heart of gravitation theory, for it is possible to argue convincingly that if **EEP** is valid, then **gravitation must be a curved-space time phenomenon**, that is, must satisfy the postulates of the Metric Theories of Gravity. These postulates state:

- Spacetime is endowed with a metric.
- The world lines of test bodies are geodesics of that metric.
- In local freely falling frames, the nongravitational laws of physics are those of special relativity.

General considerations

Gravity = Accelaration
Acceleration = Curvature

⇒ *Gravity = Curvature*



ρ = radius of curvature

n-t Coordinates move with particle.

$a_n = v^2 / \rho$

- Points toward center of curvature
- Changes direction of velocity vector

$a_t = \dot{v} = dv/dt$ along path

- Changes length of velocity vector

General considerations

Einstein Equivalence Principe (**EEP**)

- It was Einstein who added the key element to **WEP** that revealed the path to GR.

- If all bodies fall with the same acceleration in an gravitational field, then to an observer in a freely falling elevator in the same gravitational field the bodies should be unaccelerated exept for possible tidal effects due to inhomogeneities in the gravitationnal field.

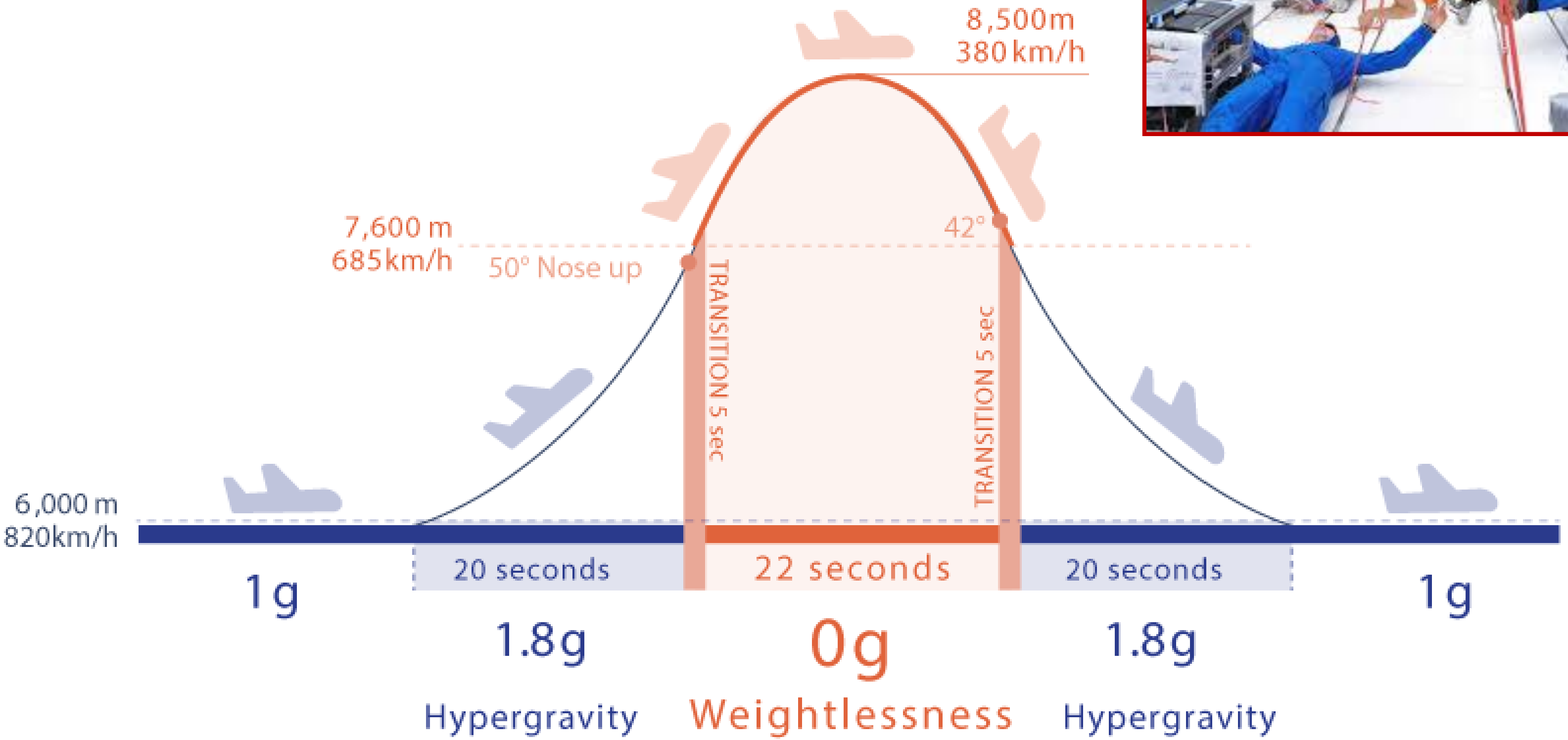
- In "The Meaning of Relativity", Einstein wrote:

“Let now K be an inertial system. Masses which are sufficiently far from each other and from other bodies are then, with respect to K, free from acceleration. We shall also refer these masses to a system of coordinates K', uniformly accelerated with respect to K. Relatively to K' all the masses have equal and parallel accelerations; with respect to K' they behave just as if a gravitational field were present and K' were unaccelerated. Overlooking for the present the question as to the "cause" of such a gravitational field, which will occupy us latter, there is nothing to prevent our conceiving this gravitational field as real, that is, the conception that K’ is "at rest" and a gravitational field is present we may consider as equivalent to the conception that only K is an "allowable" system of coordinates and no gravitational field is present. The assumption of the complete physical equivalence of the systems of coordinates, K and K', we call the "principle of equivalence" this principle is evidently intimately connected with the law of the equality between the inert and the gravitational mass, and signifies an extension of the principle of relativity to coordinate systems which are non-uniform motion relatively to each other.”

General considerations

Einstein Equivalence Principe (**EEP**)

Parabolic flights



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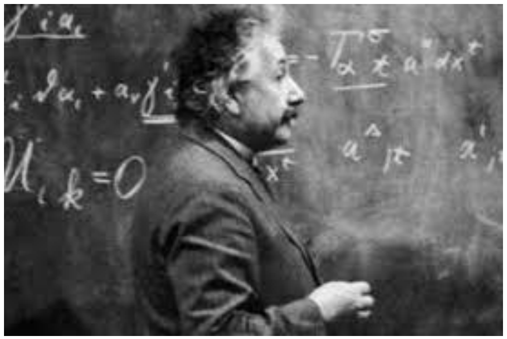

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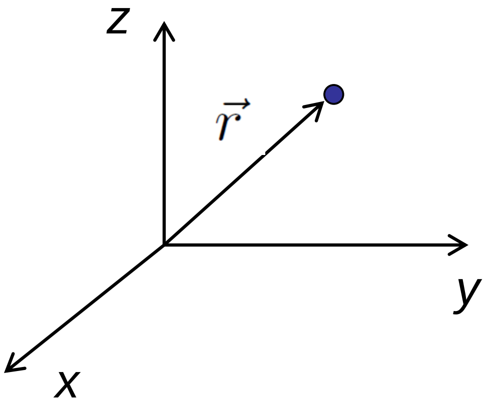
Degrees of freedom

1) Degrees of freedom (discrete number)

The position of a material point in space → $\vec{r}$



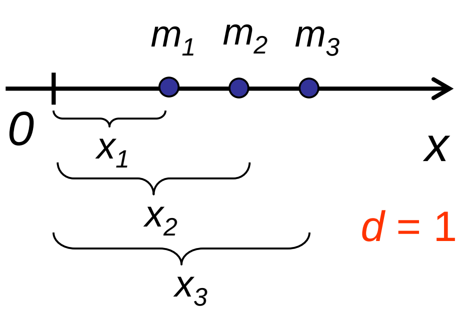
$$\vec{r}(x,y,z) \;\; ; \;\; \frac{d\vec{r}}{dt}(\dot{x},\dot{y},\dot{z}) \;\; ; \;\; \frac{d^2\vec{r}}{dt^2}(\ddot{x},\ddot{y},\ddot{z})$$

velocity acceleration

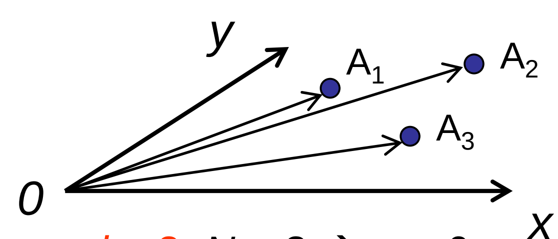
$$\dot{x} \equiv \frac{dx}{dt} \;\; ; \;\; \ddot{x} \equiv \frac{d^2x}{dt^2}$$

Definition: the number of independent quantities required to determine univocally the position of a system is called the number ***s*** of degrees of freedom (DL) of the system

- for *N* material points in *d*-dimensional space we need *N* radius vectors, i.e. *d* x *N* coordinates→ ***s*** = ***d*N***



d = 1; *N* = 3 → ***s*** = 3



d = 2; *N* = 3 → ***s*** = 6

$$\vec{OA}_1 = (x_1, y_1); \vec{OA}_2 = (x_2, y_2); \vec{OA}_3 = (x_3, y_3)$$

Degrees of freedom

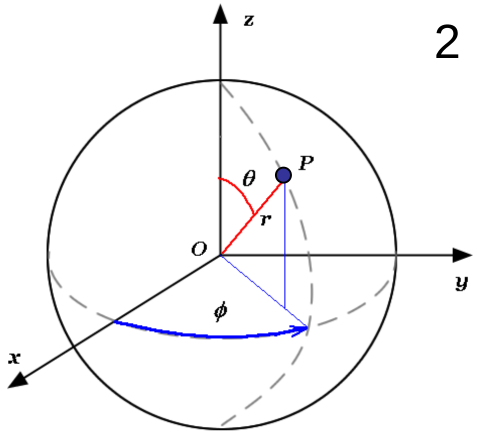
- The DLs are not necessarily the Cartesian coordinates of the point!
- It is often more convenient to use another coordinate system

$\Rightarrow q_1, q_2, \ldots q_s$
- These are the generalized coordinates $\{q\}$

fully characterize the system's position

$\Rightarrow \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s$
- These are the generalized velocities $\{\dot{q}\}$

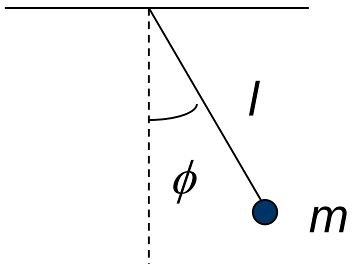
Examples:



a) Material point moving on a sphere of radius *r*

2 DL → s = 2 → (θ,ϕ) -spherical pendulum-

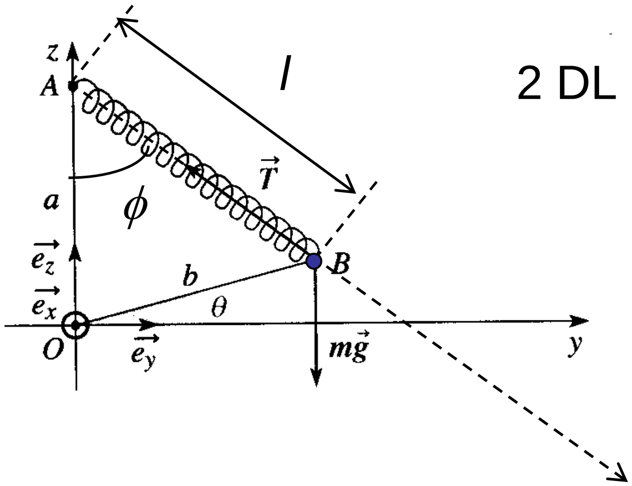
b) Planar pendulum



1DL → s = 1 → ϕ

Degrees of freedom

c)

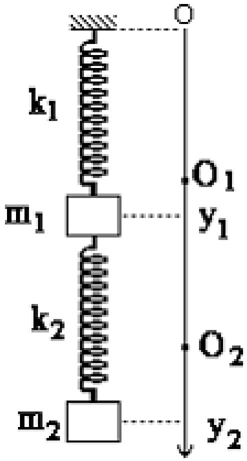


2 DL → s = 2 → (l,ϕ) ou (y_B, z_B)

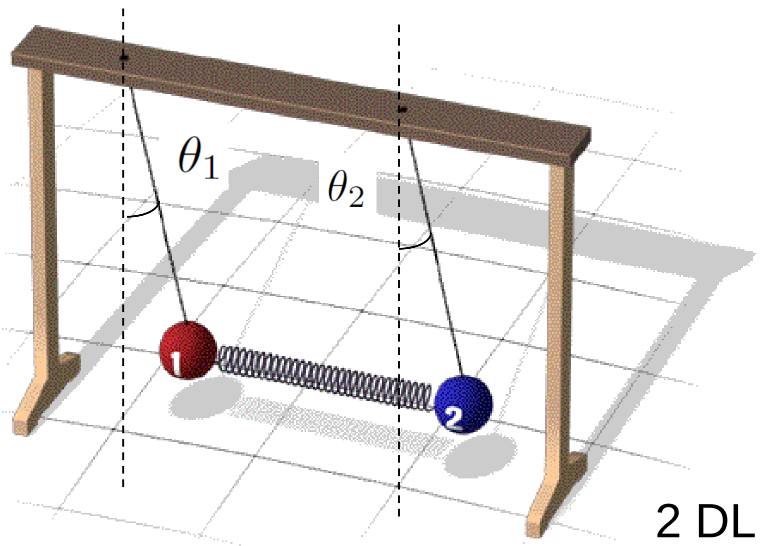
more "physical" parameterization

e) Mass-spring system

2 DL → s = 2 → (y_1,y_2)



d) Coupled planar pendulums



2 DL → s = 2 → (θ_1, θ_2)

Degrees of freedom

Experience shows that: knowledge of **coordinates and velocities** determines the state of the system and allows us to predict its final motion.

Definition: the relationships linking accelerations to coordinates and velocities are called equations of motion.

These are second-order differential equations.

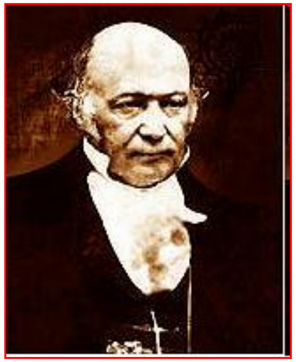
Examples:

a) $\ddot{x} + \omega_0^2 x = 0$ 1 DL

b) $\begin{cases} \ddot{x} + \omega_0^2 x = \alpha y \\ \ddot{y} + \omega_0^2 y = \alpha x \end{cases}$ 2 DL

Principle of least action

2) Principle of least action



(1805 – 1865)

It's the **Hamilton** principle

Every mechanical system is characterized by a defined function:

$$L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s, t) \equiv L(\{q\}, \{\dot{q}\}, t)$$

At times t_1 and t_2 the system occupies specific positions $\{q^{(1)}\}$ et $\{q^{(2)}\}$
Between these two instants, the system moves in such a way that the integral

$$S = \int_{t_1}^{t_2} L(\{q\}, \{\dot{q}\}, t)dt$$

has the <u>smallest</u> possible value.

L: Lagrange function of the system or Lagrangian [L] = E

S: **action** of the system [S] = E.T

$\{q\}, \{\dot{q}\}$ The **2*s*** quantities constitute a set of **dynamical variables**

Lagrange equations

- L does not contain higher derivatives $\{\not\dddot{q}\}, \{\not\ddot{q}\}, \ldots$

  (*i.e* motion is completely determined by coordinates and velocities)

- Using variational calculus, we obtain:

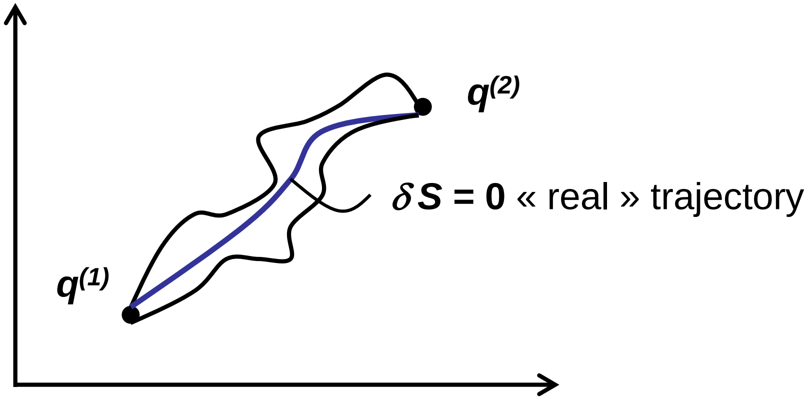
$$\begin{cases} \dfrac{\partial L}{\partial q} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = 0 & \text{1 DL} \\ \dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) = 0 \ (i = 1, \ldots, s) & \textit{s} \text{ DL} \end{cases}$$

These are the **Lagrange** equations.

It is a system of ***s*** second-order differential equations $\{\ddot{q}\}$ with ***s*** unknown functions $q_i(t)$.

Ex: ***s****=2*

$$\begin{cases} \ddot{x} + \omega_0^2 x = \alpha y \\ \ddot{y} + \omega_0^2 y = \alpha x \end{cases}$$



Galileo's principle of relativity

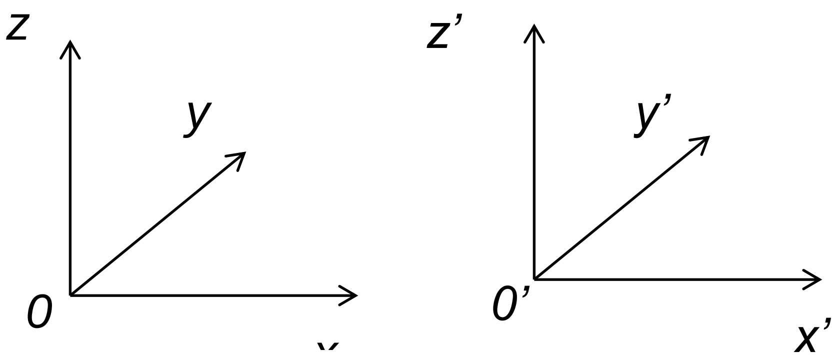
- In an IFR (Inertial frame of reference), all free movement takes place at a constant speed in magnitude and direction. This is the **law of inertia**.

- **Galileo's principle of relativity** (one of the most important principles of mechanics!)

  Let's consider two IFRs translating rectilinearly and uniformly with respect to each other. In these systems, the properties of space and time are the same, as are all the laws of mechanics

- **Galileo's transformations**

$$\begin{cases} t' = t \\ \vec{r}' = \vec{r} - \vec{v}t \end{cases}$$



$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \quad \xrightarrow[\vec{v}]{} \quad \frac{\partial L'}{\partial q'} - \frac{d}{dt'}\left(\frac{\partial L'}{\partial \dot{q}'}\right) = 0$$

Lagrange equations are invariant / to these transformations
→ Covariance

Closed system of material points

Lagrange function of a closed system of material points

➢ Consider a system of material points interacting with each other, but isolated from any foreign body; such a system is said to be **closed** (isolated).

$$L = \sum_a \frac{mv_a^2}{2} - V(\vec{r}_1, \vec{r}_2, ...) \equiv T - V$$

T is the system's kinetic energy and *V* is its potential energy.

- *V* depends only on the distribution of material points at the same time.
→ **the interaction propagates instantaneously** (only true in GaR / # in GR Einstein)
- Lagrange equations are reversible in time (*t* → -*t*)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_a}\right) = \frac{\partial L}{\partial \vec{r}_a} \quad \rightarrow \quad m_a \frac{d^2 \vec{r}_a}{dt^2} = -\frac{\partial V}{\partial \vec{r}_a}$$

Newton's equations

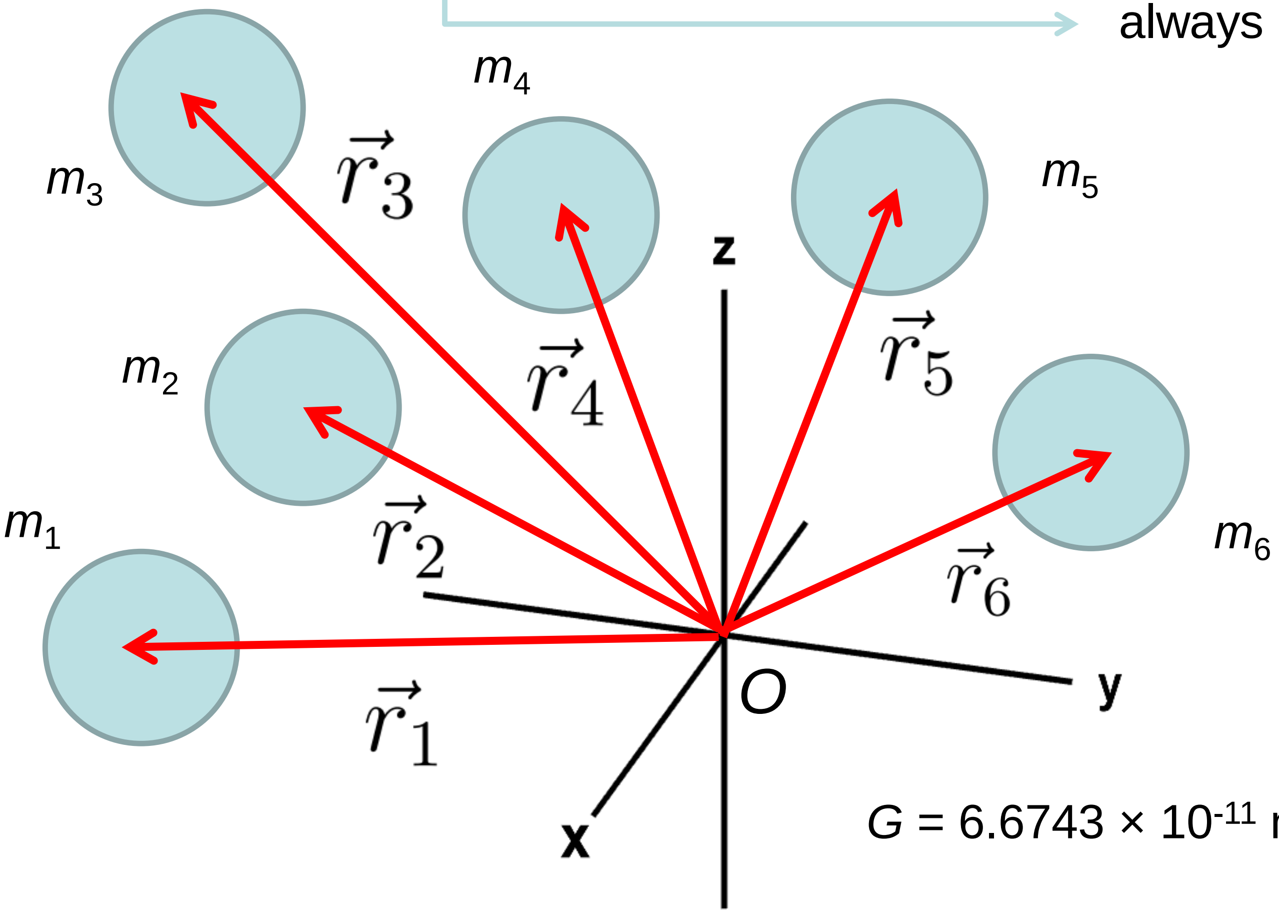
$$\vec{F}_a = -\frac{\partial V}{\partial \vec{r}_a}$$

Force acting on the point *a*

N-body problem in gravitation

$$V(\vec{r}_1, \ldots, \vec{r}_6) = -G \times \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1, j\neq i}^{6} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

always attractive



$G = 6.6743 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$

Lagrangian in generalized coordinates

$$x_a = f_a(q_1, q_2, ..., q_s), \dot{x}_a = \sum_k \frac{\partial f_a}{\partial q_k}\dot{q}_k$$

(*)

$$L = \frac{1}{2}\sum_a m_a(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2) - V \to \frac{1}{2}\sum_{ik} a_{ik}(q)\dot{q}_i\dot{q}_k - V(q)$$

Examples:

a) Cylindrical coordinates

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right) - V(r, \varphi, z)$$

b) Spherical coordinates

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - V(r, \theta, \varphi)$$

(*) **metric tensor:** The dynamical system can be associated with a **Riemannian space**

Noether's theorem

- Any infinitesimal transformation that leaves the action integral invariant corresponds to a quantity that remains constant! Symmetry → conserved quantity

Let $q_i^{(\varepsilon)}$ be generalized coordinates that depend continuously on a parameter ε. If the Lagrangian of the physical system L is independent of ε i.e. $L(q_i^{(\varepsilon)}, \dot{q}_i^{(\varepsilon)}, t) = L(q_i^{(0)}, \dot{q}_i^{(0)}, t)$ then the quantity $I(q_i, \dot{q}_i) = \frac{\partial L}{\partial \dot{q}_i} \frac{dq_i^{(\varepsilon)}}{d\varepsilon} |_{\varepsilon=0}$ is a constant of motion (or equivalent an integral of motion).

(1882-1935)

Noether's theorem

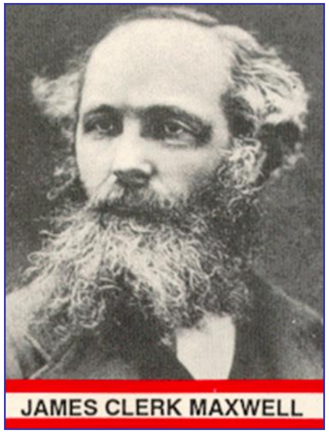
External space-time symmetries
→ Space and time translations, space rotation

- **Energy**: time translation invariance
- **Linear momentum**: space translation invariance
- **Angular momentum**: space rotation invariance

Note: There are also "**internal**" symmetries (gauge).

Concept of field

Before **James Clerk Maxwell**



(1831-1879)

- In the thinking of Coulomb, Ampère and Faraday, inherited from Newton we only know the forces exerted by one material system on another (the notion of "material system" includes electrical matter or fluid capable of exerting or undergoing such forces).

- The notion of **FIELD** (electric or magnetic), which is widely used (notably by Faraday) has no other meaning than that of a virtual force that only becomes real when and where a charged body, or a magnet, or an electric current element is present.

- Ultimately, the field is no more than a convenient calculation intermediary...

The problem (since Newton) of instantaneous action at a distance...

Concept of field

« …With **J. Clerk Maxwell**, a new scientific scientific era opened… »
Albert Einstein

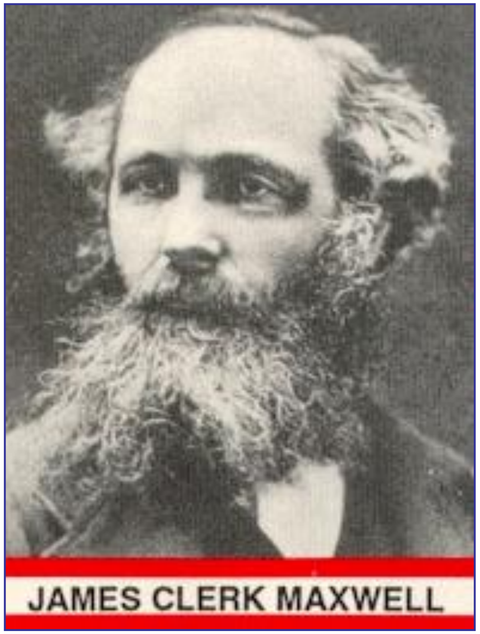
$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} . \vec{B} = 0$$

$$\vec{\nabla} . \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

(1873)



« Imagine his emotion when the equations he had formulated revealed that electromagnetic fields propagate as polarized waves, and at the speed of light! Few men in the world can claim to have had such an experience. » **Albert Einstein**

Concept of field

Maxwell's equations in vacuum

The purpose of Maxwell's equations is to link the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ to their sources, charge and current densities $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$ in the general case of variable regimes.

$$\overrightarrow{\nabla} \wedge \overrightarrow{B} = \mu_0 \overrightarrow{j} + \varepsilon_0 \mu_0 \frac{\partial \overrightarrow{E}}{\partial t}$$

Maxwell-Ampère

Vector field (tensor of rank 1; one indice ν)

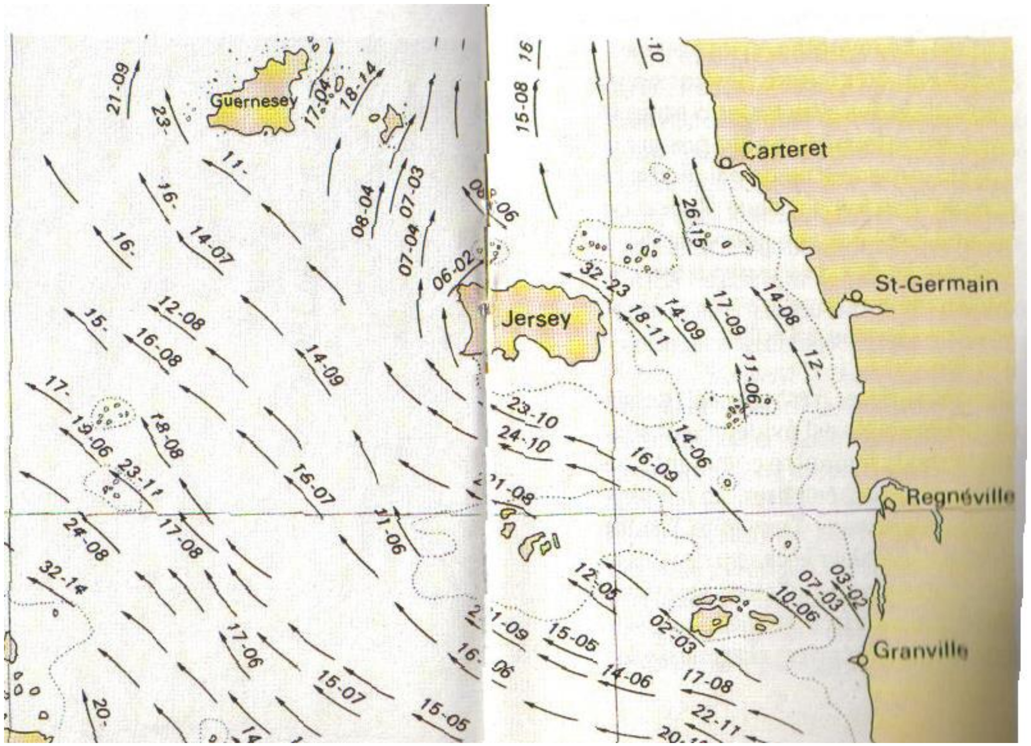
In the case of variable regimes, the fields $\vec{E}(\vec{r},t) \equiv E_\nu(\vec{r},t)$ and $\vec{B}(\vec{r},t) \equiv B_\nu(\vec{r},t)$ are strongly coupled and form an entity to which we give the name electromagnetic field $F_{\mu\nu}(\vec{r},t)$.

Tensor field (tensor of rank 2; two indices ν and ν)

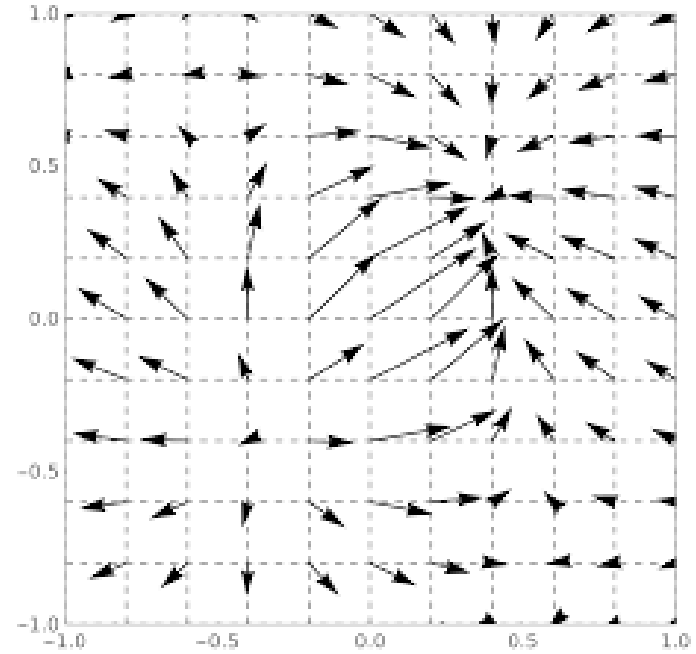
Maxwell's equations are the very expression of the fundamental laws of electromagnetism. It's an extremely concise, aesthetic, compact and elegant formulation that describes a vast class of phenomena, including optics, radio-electricity, guided waves...

- Formulation in terms of partial differential equations.
- This leads naturally to the idea of a finite speed of propagation and wave equation (*em* waves).
- For **Newton**, the (continuous) space between two bodies was irrelevant.

Concept of field



Tidal currents map



Vector field

Maxwell gives birth to a new being in mathematical physics:

The vector field!

Continuous number of degrees of freedom

Continuous number of degrees of freedom (application to *em*)

- The state of the system is now determined by a **continuous** rather than **discrete** number of dynamical variables. This extension is necessary insofar as we want to study a **vector field** that is defined by its value at all points in space. We will now consider generalized coordinates that depend on a **continuous index** (denoted by $\vec{r}$ the current point in 3-dimensional space) and a discrete index *j* (*j* varying from 1 to *N*).

- As in the discrete case, the generalized coordinates and velocities are defined by

$$q_n(t) \rightarrow A_j(\vec{r}, t)$$

$$\dot{q}_n(t) \rightarrow \dot{A}_j(\vec{r}, t) \equiv \frac{\partial A_j}{\partial t}$$

and constitute a set of dynamical variables for the system.

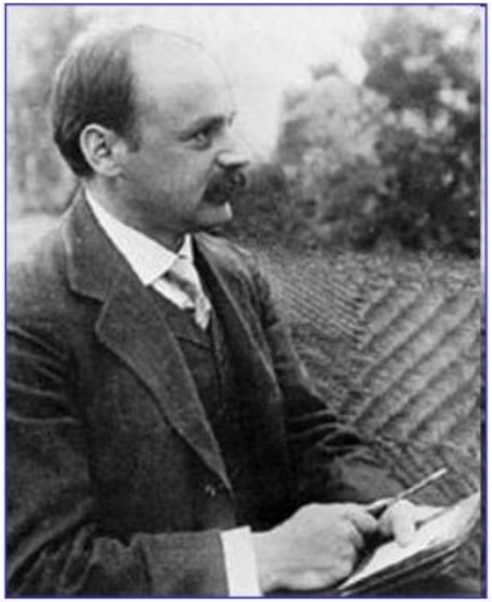
$$A_j(\vec{r}, t)$$

discret index
vector components

continous index

Principle of least action for the field

- The Lagrangian formulation of classical electrodynamics based on the principle of least action was realized by **Karl Schwarzschild** in 1903.



(1873-1916)

- Schwarzschild provided **the first exact solution to the Einstein field equations of general relativity**, for the limited case of a single spherical non-rotating mass, which he accomplished in 1915, the same year that Einstein first introduced general relativity. The Schwarzschild solution, which makes use of Schwarzschild coordinates and the Schwarzschild metric, leads to a derivation of the Schwarzschild radius, which is the size of the event horizon of a non-rotating **black hole**.

Principle of least action for the field

$$S = \int_{t_1}^{t_2} dt \int \mathcal{L}(A_j, \dot{A}_j, \partial_i A_j, t) d\vec{r}$$

$$\partial_i = \partial x, \partial y, \partial z$$

- $\mathcal{L}(A_j, \dot{A}_j, \partial_i A_j, t)$ is the lagrangian density.
- The term $\partial_i A_j$ is necessary for describing the Maxwell equations that are non-local (the evolution of the coordinate $A_j(\vec{r}, t)$ is coupled with the evolution of the coordinate of a neighbor point).
- In the same way that, in a problem with discrete variables q_j, the evolution of q_j can depend on q_{j-1} and q_{j+1} (see the demonstration on the example of the chain of coupled harmonic oscillators).
- The principle of least action applied to S leads to the Lagrange equations for the fields.

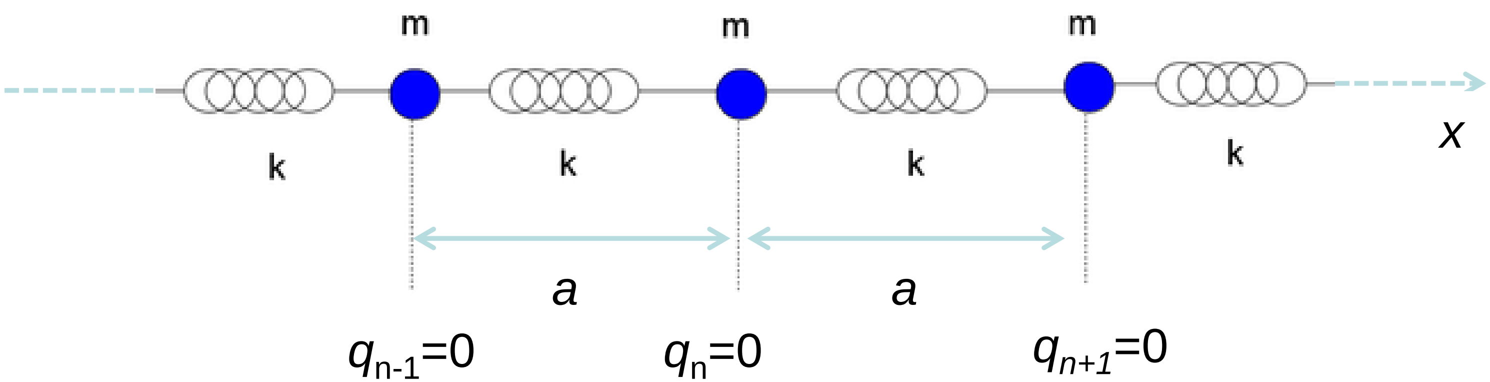
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{A}_j} = \frac{\partial \mathcal{L}}{\partial A_j} - \sum_{i=x,y,z} \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)}$$

NEW

Discrete to continuous: scalar field

Chain of coupled identical harmonic oscillators

The aim of what follows is to intuitively introduce the concept of Lagrangian and Hamiltonian densities of a continuous system by studying how a discrete mechanical system can be transformed into a continuous system.



Discrete to continuous

Discrete system

Consider an infinite set of point particles of mass m aligned along the x axis with equilibrium spacing a (see Figure). The displacement along the x axis of the n^{th} particle (whose equilibrium position is na) is called q_n. The state of the system at time t is fixed by giving the dynamical variables $q_n(t)$ and $\dot{q}_n(t)$. The potential energy of the system of n particles depends on their separations and is equal to

$$V = \frac{1}{2} m\omega_1^2 \sum_n (q_{n+1} - q_n)^2 \ , \tag{1}$$

with $\omega_1^2 = k/m$. The Lagrangian of this system of point particles is

$$L = \sum_n \frac{1}{2} m\dot{q}_n^2 - \sum_n \frac{1}{2} m\omega_1^2 (q_{n+1} - q_n)^2. \tag{2}$$

The relations

$$\frac{\partial L}{\partial \dot{q}_n} = m\dot{q}_n \tag{3}$$

and

$$\frac{\partial L}{\partial q_n} = m\omega_1^2 (q_{n+1} - q_n) - m\omega_1^2 (q_n - q_{n-1}) \tag{4}$$

lead to the equations of motion

$$\ddot{q}_n = \omega_1^2 (q_{n+1} - q_n) - \omega_1^2 (q_n - q_{n-1}) \ . \tag{5}$$

Discrete to continuous

Let's look for a solution of the form $q_n(t) = \delta e^{i(kna-\omega t)}$. This form is solution of the above equation if

$$-\omega^2 = \omega_1^2[(e^{ika}-1)-(1-e^{-ika})] = -4\omega_1^2 \sin^2\left(\frac{ka}{2}\right) . \tag{6}$$

We calculate the phase velocity $v = \omega/k$ and its limit when $k \to 0$, v_0. We have $v = \frac{2\omega_1}{k}\sin\left(\frac{ka}{2}\right)$ and $v_0 = \omega_1 a$. By using the results of the course on analytical mechanics, one gets immediately, $p_n = \frac{\partial L}{\partial \dot{q}_n} = m\dot{q}_n$ and

$$H = \sum_n p_n \dot{q}_n - L = \sum_n \frac{1}{2} m \dot{q}_n^2 + \sum_n \frac{1}{2} m \omega_1^2 (q_{n+1}-q_n)^2 . \tag{7}$$

Discrete to continuous

The continuous system gotten by passing to the limit

Let the distance a between two adjacent particles and the mass m of each particle go to zero in such a way that the mass per unit length, $\mu = m/a$, is kept constant. Similarly, let ω_1 vary in such a way that when a goes to 0, v_0 remains constant. One gets then in this limit a continuous string with mass per unit length μ and where the velocity of sound is v_0.

The discrete dynamical variable $q_n(t)$ which represents the displacement of the point na becomes a continuous variable $q(x,t)$ giving the displacement of a point on the string whose equilibrium position is x. Similarly, $\dot{q}_n(t)$ becomes $\partial q(x,t)/\partial t$ in the continuous limit. One then moves from a discrete index n to a continuous index x.

The Lagrangian (2) can be written

$$L = \mu \sum_n a \left[\frac{\dot{q}_n^2}{2} - \frac{v_0^2}{2} \frac{(q_{n+1} - q_n)^2}{a^2} \right] , \tag{8}$$

since $\mu = m/a$ and $v_0 = \omega_1 a$.

When a goes to zero, $(q_{n+1} - q_n)/a$ tends to $\partial q(x)/\partial x$ and the above expression of L becomes

$$L = \int dx \frac{\mu}{2} \left[(\dot{q}(x))^2 - v_0^2 \left(\frac{\partial q(x)}{\partial x} \right)^2 \right] . \tag{9}$$

Discrete to continuous

The Lagrangian density $\mathcal{L}$ is then given by

$q(x,t)$ is a scalar field in one dimension

$$\mathcal{L} = \frac{\mu}{2}\left[(\dot{q}(x))^2 - v_0^2\left(\frac{\partial q(x)}{\partial x}\right)^2\right] . \tag{10}$$

The equation of motion (5) can be rewritten in the form

$$\ddot{q}_n = \omega_1^2 a^2 \frac{[(q_{n+1} - q_n)/a] - [(q_n - q_{n-1})/a]}{a} . \tag{11}$$

When $a \to 0$,

$$(q_{n+1} - q_n)/a \to \frac{\partial q(x)}{\partial x} \tag{12}$$

$$(q_n - q_{n-1})/a \to \frac{\partial q(x-a)}{\partial x} \tag{13}$$

and thus Equation (11) tends to

$$\ddot{q}(x) = v_0^2 \frac{\partial^2 q(x)}{\partial x^2} . \tag{14}$$

This expression can also be found by using (10) and Lagrange's equation for the field

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{A}_j} = \frac{\partial \mathcal{L}}{\partial A_j} - \sum_{i=x,y,z} \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)} . \tag{15}$$

Discrete to continuous

In fact we have: (i) one-dimensional model i.e. $\vec{r} \to x$; (ii) $q(x,t)$ is a scalar field which corresponds to one component of the vectorial field $A_j(\vec{r},t)$ and $i = x$; (iii) $\frac{\partial \mathcal{L}}{\partial \dot{q}} = \mu \dot{q}$; (iv) $\frac{\partial \mathcal{L}}{\partial q} = 0$; (vi) $-\sum_i \to -\partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x q)} = -\partial_x \left(\frac{\partial q}{\partial x}\right) \times (-\mu v_0^2) = \mu v_0^2 \frac{\partial^2 q}{\partial x^2}$.

Equation (14) is a wave equation. Using (10) one gets $\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}(x)} = \mu \dot{q}(x)$ which corresponds to the limit of p_n/a when a tends to 0. The Hamiltonian H of the discrete system $H = \sum_n p_n \dot{q}_n - L = a \sum_n \frac{p_n}{a} \dot{q}_n - L$ has as its limit when $a \to 0$

$$H = \int dx \mathcal{H} \ , \tag{16}$$

with

$$\mathcal{H} = \Pi(x)\dot{q}(x) - \mathcal{L} \tag{17}$$

so that, using (10)

$$\mathcal{H} = \frac{\Pi^2(x)}{2\mu} + \mu \frac{v_0^2}{2} \left(\frac{\partial q(x)}{\partial x}\right) \ . \tag{18}$$

Scalar field theory

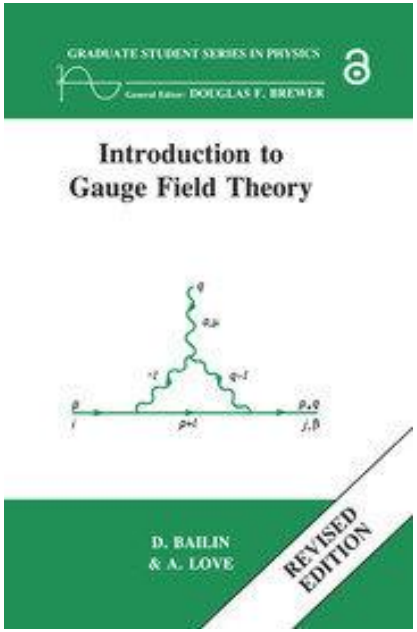
One of the simplest field theory is described by a single real scalar field $\varphi(x)$ having a lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \,, \qquad (1)$$

where μ^2 and λ are constants. Find the Euler-Lagrange equation. Answer:

$$\left(\partial_\mu\partial^\mu + \mu^2\right)\varphi(x) = -\frac{1}{6}\lambda\varphi^3(x) \,. \qquad (2)$$

- **Noether**'s theorem: Bailin & Love 3.2, page 18



Lagrangian for *em* field

Non-relativistic (NR) lagrangian of the system « fields + particles »

$$L \equiv \underbrace{\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}}_{L_P \text{ (Particles)}} + \underbrace{\frac{\epsilon_0}{2} \int \left(\vec{E}^2 - c^2 \vec{B}^2 \right) d\vec{r}}_{L_R \text{ (Fields)}} + \underbrace{\sum_i q_i \dot{\vec{r}}_i . \vec{A}(\vec{r}_i) - q_i U(\vec{r}_i)}_{L_I \text{ (Interaction particles-fields)}}$$

with $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} U$

- L_P NR lagrangian for the particles (only kinetic energy). Discrete number of charged particles characterized by (m_i, q_i).
- L_R Lagrangian for the fields (electric + magnetic).
- L_I Lagrangian describing the interactions between the fields and the particles.
- In the following $d\vec{r} = dxdydz = dV = d^3\vec{r}$.

Lagrangian for *em* field

$$L_I \equiv \int \left[\vec{j}(\vec{r}).\vec{A}(\vec{r}) - \rho(\vec{r})U(\vec{r})\right] d\vec{r} \equiv \int \mathcal{L}_I \, d\vec{r}$$

$$\boxed{j_\mu A^\mu}$$

relativistic notation (later)

- In the following we will show that this lagrangian leads to the Maxwell-Lorentz equations.
- $\mathcal{L}_I$ denotes de lagrangian density.
- The generalized variables are $\left\{ (\vec{r}_i)_j \, , \left(\dot{\vec{r}}_i\right)_j \right\}$, $\left\{ A_j(\vec{r}), \dot{A}_j(\vec{r}) \right\}$ and $\left\{ U(\vec{r}), \dot{U}(\vec{r}) \right\}$.
- We have $\rho(\vec{r}) = \sum_i q_i \delta(\vec{r} - \vec{r}_i)$ and $\vec{j}(\vec{r}) = \sum_i q_i \dot{\vec{r}}_i \delta(\vec{r} - \vec{r}_i)$.

$(\vec{r}_i)_j$

- j: j^{th} component of the i^{th} particle
- i: i^{th} particle

Gauge invariance – internal symmetry

- L_R is not modified since it only involves the fields $\vec{E}$ and $\vec{B}$ that by definition do no change under a gauge transformation.

- The lagrangian of the particles is not modified.

- Only the interaction lagrangian is modified.

$$L_I = \int \left[\vec{j}(\vec{r}).\vec{A}(\vec{r}) - \rho(\vec{r})U(\vec{r})\right] d\vec{r} = \int \mathcal{L}_I d\vec{r}$$

$$\begin{cases} \vec{A}(\vec{r},t) &= \vec{A}_0(\vec{r},t) + \vec{\nabla}\phi(\vec{r},t) \\ U(\vec{r},t) &= U_0(\vec{r},t) - \dfrac{\partial\phi(\vec{r},t)}{\partial t} \end{cases} \Rightarrow \quad \mathcal{L}'_I = \mathcal{L}_I + \mathcal{L}_1$$

$$\mathcal{L}_1 = \vec{j}.\vec{\nabla}\phi + \rho\frac{\partial\phi}{\partial t} = \underbrace{\vec{\nabla}.(\vec{j}\phi)}_{(1)} + \underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{(2)} - \underbrace{\boxed{\left(\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t}\right)}}_{(3)}\phi \quad (*)$$

= 0

(1) If one integrates over the whole space, the term with the divergence vanishes
(2) A time derivative does not change the equations of motion
(3) Finally, the **charge conservation** is a necessary condition for the gauge invariance
(This is in agreement with the **Noether**'s theorem)

(*) $\vec{\nabla}.(\vec{j}\phi) = (\vec{\nabla}.\vec{j})\phi + \vec{j}.\vec{\nabla}\phi$

Gauge invariance – internal symmetry

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Historical roots of gauge invariance

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Gauge invariance is the basis of the modern theory of electroweak and strong interactions (the so-called standard model). A number of authors have discussed the ideas and history of quantum guage theories, beginning with the 1920s, but the roots of gauge invariance go back to the year 1820 when electromagnetism was discovered and the first electrodynamic theory was proposed. We describe the 19th century developments that led to the discovery that different forms of the vector potential (differing by the gradient of a scalar function) are physically equivalent, if accompanied by a change in the scalar potential: $\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}+\nabla\chi$, $\Phi\rightarrow\Phi'=\Phi-\partial\chi/c\partial t$. L. V. Lorenz proposed the condition $\partial_\mu A^\mu=0$ in the mid-1860s, but this constraint is generally misattributed to the better known H. A. Lorentz. In the work in 1926 on the relativistic wave equation for a charged spinless particle in an electromagnetic field by Schrödinger, Klein, and Fock, it was Fock who discovered the invariance of the equation under the above changes in $\mathbf{A}$ and Φ if the wave function was transformed according to $\psi\rightarrow\psi'=\psi\exp(ie\chi/\hbar c)$. In 1929, H. Weyl proclaimed this invariance as a general principle and called it *Eichinvarianz* in German and *gauge invariance* in English. The present era of non-Abelian gauge theories started in 1954 with the paper by Yang and Mills on isospin gauge invariance.

Lorentz equation

- The **Lorentz** equation is obtained with the particle dynamic variables

From the Lagrange equations for the particles

$$\frac{d}{dt}\frac{\partial L}{\partial(\dot{\vec{r}}_i)_j} = \frac{\partial L}{\partial(\vec{r}_i)_j}$$

$$\left\{ (\vec{r}_i)_j ; (\dot{\vec{r}}_i)_j \right\}$$

one obtains the Lorentz equation

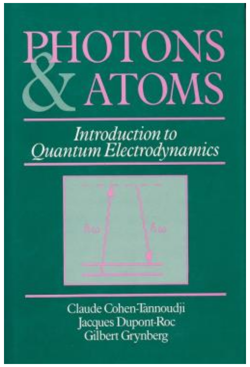
$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \underbrace{q_i \vec{E}(\vec{r}_i) + q_i \dot{\vec{r}}_i \times \vec{B}(\vec{r}_i)}_{\textbf{Lorentz force}}$$

Lorentz force

Note: In relativistic mechanics $\vec{p} = \frac{m\dot{\vec{r}}}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\vec{p} \to m\vec{v}$ when $v \ll c$ i.e. in the non-relativistic limit.

Thus we have (for one particle)

$$\frac{d\vec{p}}{dt} = q\vec{E}(\vec{r}) + q\dot{\vec{r}} \times \vec{B}(\vec{r})$$



Proof in this book

Poisson and Ampère equations

$$L \equiv \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 + \frac{\epsilon_0}{2} \int \left(\vec{E}^2 - c^2 \vec{B}^2 \right) d\vec{r} + \sum_i q_i \dot{\vec{r}}_i . \vec{A}(\vec{r}_i) - q_i U(\vec{r}_i)$$

- Replace $\vec{E}$ and $\vec{B}$ by $\vec{E} = -\dot{\vec{A}} - \vec{\nabla} U$ and $B = \vec{\nabla} \times \vec{B}$ in the lagrangian density.
- The Lagrange equation relative to U is found by evaluating $\frac{\partial \mathcal{L}}{\partial \dot{U}}$, $\frac{\partial \mathcal{L}}{\partial U}$ and $\frac{\partial \mathcal{L}}{\partial (\partial_i U)}$.
- We have: $\frac{\partial \mathcal{L}}{\partial \dot{U}} = 0$, $\frac{\partial \mathcal{L}}{\partial U} = -\rho$ and $\frac{\partial \mathcal{L}}{\partial (\partial_i U)} = \frac{\partial \mathcal{L}}{\partial E_i} \frac{\partial E_i}{\partial (\partial_i U)} = -\varepsilon_0 E_i$.
- From which one gets Lagrange equation $-\rho + \varepsilon_0 \sum_i \partial_i E_i = 0$ that is $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$.
- Ampère's equation $\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \dot{\vec{E}} + \frac{1}{\varepsilon_0 c^2} \vec{j}$ is obtained from the dynamical variables A_j.

Noether's theorem for the *em* field

- **a) Energy**: time translation invariance
- **b) Linear momentum**: space translation invariance
- **c) Angular momentum**: space rotation invariance

a) Time

invariance of the **Maxwell-Lorentz** equations under a change of the time origin is related to the total energy conservation

$$H = \underbrace{\sum_i \frac{1}{2} m_i v_i^2(t)}_{\text{Kinetic energy of the charged particles}} + \underbrace{\frac{\epsilon_0}{2} \int \left(\|\vec{E}(\vec{r},t)\|^2 + c^2 \|\vec{B}(\vec{r},t)\|^2 \right) d\vec{r}}_{\text{Field energy}}$$

- The Coulomb interaction among the charged particles is included in the field energy

Noether's theorem for the *em* field

- **a) Energy**: time translation invariance
- **b) Linear momentum**: space translation invariance
- **c) Angular momentum**: space rotation invariance

b) Space-translation

invariance of the **Maxwell-Lorentz** equations under a change of the origin of the spatial coordinate system is related to the conservation of the total linear momentum

$$\vec{P} = \underbrace{\sum_i m_i \vec{v}_i(t)}_{\text{Linear momentum of the charged particles}} + \epsilon_0 \underbrace{\int \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)\, d\vec{r}}_{\text{Field linear momentum}}$$

Linear momentum of the charged particles — Field linear momentum

Laser cooling (BEC, Radiation pressure…)

Noether's theorem for the *em* field

- **a) Energy**: time translation invariance
- **b) Linear momentum**: space translation invariance
- **c) Angular momentum**: space rotation invariance

c) Space-rotation

invariance of the **Maxwell-Lorentz** equations under a change of the orientation of the axis of the spatial coordinate system is related to the conservation of the total angular momentum

$$\vec{J} = \underbrace{\sum_i \vec{r}_i(t) \times m_i \vec{v}_i(t)}_{\text{Angular momentum of the particles}} + \epsilon_0 \underbrace{\int \vec{r} \times \left(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right) \, d\vec{r}}_{\text{Field angular momentum}}$$

Angular momentum of the particles

Field angular momentum

- **Noether**'s theorem for **continuous fields** in **four-dimensional space–time** corresponds to the conservation law for the **stress–energy tensor** (crucial for GR).

Euler-Lagrange (most general)

Given a field tensor ϕ, a scalar, called the Lagrangian density $\mathcal{L}(\phi, \partial\phi, \partial\partial\phi, \ldots, x)$ can be constructed from ϕ and its derivatives. From this density, the action functional can be constructed by integrating over the space-time,

$$\mathcal{S} = \int \mathcal{L}\sqrt{-g}\,\mathrm{d}^4x \ ,$$

where $\sqrt{-g}\,\mathrm{d}^4x$ is the volume form in curved space-time and $g \equiv \det(g_{\mu\nu})$.

Then by using the principle of least action the Euler-Lagrange equations can be obtained

$$\frac{\delta\mathcal{S}}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \cdots + (-1)^m\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{m-1}}\partial_{\mu_m}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{m-1}}\partial_{\mu_m}\phi)}\right) = 0 \ .$$

1. Generalized with higher derivatives of the field (*em*: only first derivative).
2. Relativistic extension (Minkowski space).
3. Generalized for curved space ($g_{\mu\nu}$). g=det($g_{\mu\nu}$).
4. In Lorentz coordinates g=-1 and d^4x=dVdt (e.g. *em*).

The action of general relativity

David Hilbert (1862-1943)

Wir müssen wissen. Wir werden wissen.
(We need to know. We will know.)

The father of the XXIII problems

On August 8, 1900, at the Second International Congress of Mathematicians in Paris, he set out the future of mathematics in the form of 23 problems to be solved by the 20th century. Five remain unsolved to this day, including...the Riemann conjecture.

The Einstein equations of the gravitational field are obtained in the same way as the Euler-Lagrange equations, by varying (with respect to the metric tensor g_{ij}) the Hilbertian action S_g of this field.

We define S_g as

$$S_g = \int R\sqrt{-g}d^4x$$

where the integral is taken on the infinite edge of three-dimensional space (x^1, x^2, x^3) and between the bounds $x_1^0 \leq x^0 \leq x_2^0$ for time x^0.

***R* is the curvature**