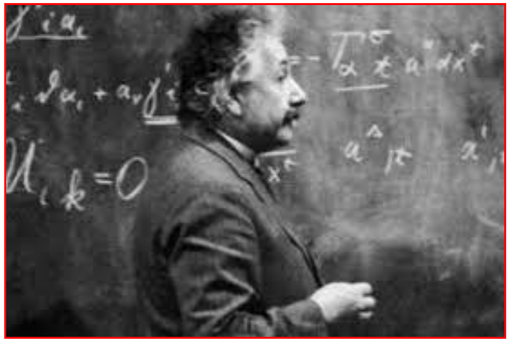
General Relativity (GR)

M1 - Physique 2024-2025



AE+GR (1907-1917)

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$$g_{\mu\nu}$$

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

II) preliminaries of physics and mathematics

- General considerations (Einstein equivalence principle…)
- Field theory
- Tensors
- Special relativity, and complements

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

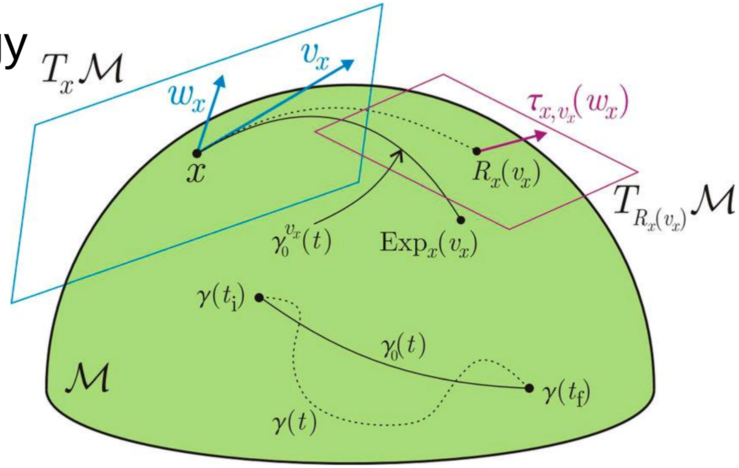
$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

Tensor calculus

Tools to be developed

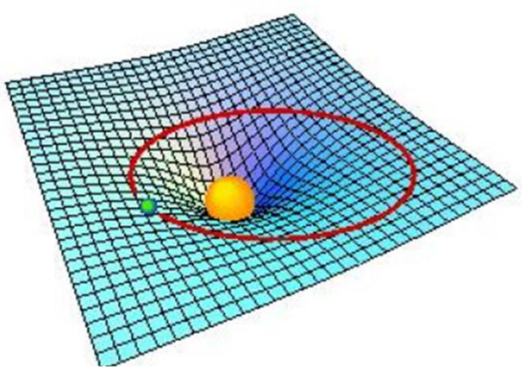
g_{ab} ← metric, $F^a_{\ b}$ ← electromagnetic, R_{ab} ← Ricci, T_{ab} ← Stress-energy : tensors

→ **Tensor calculus**



Γ^a_{bc} → Levi-Civita **connection**

R_{ab} → **Riemannian geometry /differential geometry**



- Gravity is the result of distortions in space-time created by mass and energy

Tensors

"*Tensor calculus knows physics better than the physicist himself*"

said Paul Langevin (one of Einstein's best friends).

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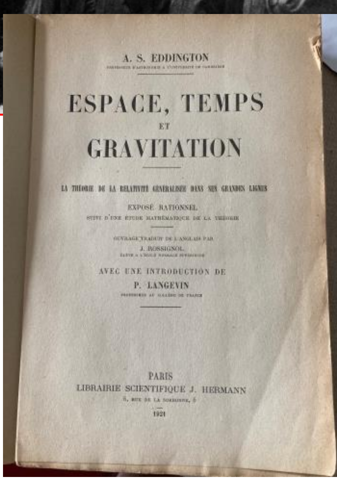
1911 Solvay conferences 1927

- Paul Langevin prefaced Arthur Edington's book


A. S. EDDINGTON

ESPACE, TEMPS
ET
GRAVITATION

LA THÉORIE DE LA RELATIVITÉ GÉNÉRALISÉE DANS SES GRANDES LIGNES

EXPOSÉ RATIONNEL
SUIVI D'UNE ÉTUDE MATHÉMATIQUE DE LA THÉORIE

OUVRAGE TRADUIT DE L'ANGLAIS PAR
J. ROSSIGNOL

AVEC UNE INTRODUCTION DE
P. LANGEVIN

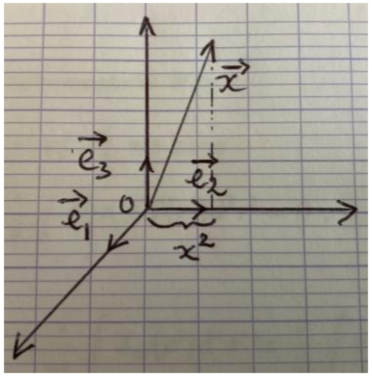
PARIS
LIBRAIRIE SCIENTIFIQUE J. HERMANN
5, RUE DE LA SORBONNE, 5
1921

Tensor calculus

I. VECTORS

A. Writing conventions

1. Notation of vectors and their components



A vector $\vec{x}$ in classical geometry, related to a basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is written as:

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \tag{1}$$

We will also use lower indices for the components (see covariant and contravariant components).

2. Convention for the summation

$$\boxed{x^1y^1 + x^2y^2 + ... + x^ny^n = \sum_{i=1}^{n} x^iy^i \equiv x^iy^i.} \tag{2}$$

Einstein's summation convention consists in using the fact that the repeated index, in this case i, will itself define the indication of the summation. More examples are given below:

- $A_{ii}x_j = A_{11}x_j + A_{22}x_j + ... + A_{nn}x_j \equiv A_{ii}x_j$
- $A^1_ky^1 + A^2_ky^2 + A^3_ky^3 + A^4_ky^4 \equiv A^i_ky^i$ for $n = 4$. In this example i is called silent indice and k free indice.

Tensor calculus

- $A_{11} + A_{22} + ... + A_{nn} \equiv A_{ii}$

- $a_{ij}x_j = b_i$ for $n = 3$ leads to

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

- $\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = x^i\vec{e}_i$

In conclusion, any expression with a twice-repeated subscript represents a sum over all possible values of the repeated subscript.

Tensor calculus

3. *Summation over several indices*

The summation convention extends to the case where there are several silent indices in the same expression. For example below for $i, j = 1, 2$:

$$
\begin{aligned}
A^i_j x^i y^j &= A^1_j x^1 y^j + A^2_j x^2 y^j \ \text{ summation over } i \\
&= A^1_1 x^1 y^1 + A^1_2 x^1 y^2 + A^2_1 x^2 y^1 + A^2_2 x^2 y^2 \ \text{ summation over } j
\end{aligned}
$$

Suppose we have the relationship $\alpha = a_{ij} x^i y_j$ with $x^i = c_{ij} y^j$. It's not right to write $\alpha = a_{ij} c_{ij} y^j y_j$. First, we need to rewrite x^i with another silent index as $x^i = c_{ik} y^k$ and then $\alpha = a_{ij} c_{ik} y^k y_j$. This results in a triple summation over the mute indices i, j, k. It leads to ($2^3 = 8$ terms, $n = 2$ and 3 indices) $a_{11} c_{11} y^1 y_1 + a_{11} c_{12} y^2 y_1 + a_{12} c_{11} y^1 y_2 + a_{12} c_{12} y^2 y_2 + a_{21} c_{21} y^1 y_1 + a_{21} c_{22} y^2 y_1 + a_{22} c_{21} y^1 y_2 + a_{22} c_{22} y^2 y_2$.

The summation can be generalized to any number of indices.

Tensor calculus

4. Kronecker symbol

This symbol is called the Kronecker symbol.

$$\boxed{\delta_{ij} = \delta_i^j = \delta^{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}} \tag{3}$$

It can be used to write, for example the scalar product of two vectors $\vec{e}_i$ and $\vec{e}_j$ of norm unity and orthogonal (also called orthonormal) to each other in the form $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. We have

$$\delta_{ij} y_i y_j = y_i y_i \ . \tag{4}$$

(1823-1891)

Note: only δ_i^j is tensorial (second-rank tensor). δ_{ij} and δ^{ij} are not tensors.

Proof: if u_i^j is a tensor we have by definition

$$\begin{aligned} u_k'^l &= A_k^i A_j'^l u_i^j \\ &= A_k^i A_j'^l \delta_i^j \\ &= A_k^j A_j'^l = \delta_k^l \ . \end{aligned}$$

$A_k^j A_j'^l$ is the product of a matrix and its inverse, which leads to the unit matrix. δ_{ij} and δ^{ij} do not verify this.

Tensor calculus

5. *Antisymmetry symbol*

In the case where the indices i, j, k take one of the values $1, 2, 3$, the antisymmetry symbol ϵ^{ijk} takes on the following values:

- $\epsilon^{ijk} = 0$ if any two of the indices have identical values. For example: $\epsilon^{112} = \epsilon^{313} = \epsilon^{222} = 0$.
- $\epsilon^{ijk} = 1$, if the indices are in the order $1, 2, 3$ or come from an even number of permutations of this initial order. For example: $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1$.
- $\epsilon^{ijk} = -1$, if the indices come from an odd number of permutations of initial order $1, 2, 3$. For example: $\epsilon^{132} = \epsilon^{321} = \epsilon^{213} = -1$.

Using this symbol, a second-order determinant can be written in the following form $\det[a^{ij}] = \epsilon^{ij} a^{1i} a^{2j}$ and a third-order determinant as $\det[a^{ij}] = \epsilon^{ijk} a^{1i} a^{2j} a^{3k}$.

Tensor calculus

B. Change of basis in a vector space

Let us consider two basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $(\vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3)$ of a vector space E_n. Each vector of one basis can be decomposed on the other basis in the following form

$$\vec{e}_i = A_i'^k \vec{e}\,'_k \ , \tag{5}$$

and

$$\vec{e}\,'_k = A_k^i \vec{e}_i \ , \tag{6}$$

where we use the summation convention for $i, k = 1, 2, ..., n$.

Change the components of a vector

A vector $\vec{x}$ of E_n can be decomposed on each basis as

$$\vec{x} = x^i \vec{e}_i = x'^k \vec{e}\,'_k \ . \tag{7}$$

Let's look for relationships between the components x^i and x'^k. Let's replace the vectors $\vec{e}_i$ and $\vec{e}\,'_k$ in the above relationship by their respective expressions. It leads to

$$\vec{x} = x^i \vec{e}_i = x^i A_i'^k \vec{e}\,'_k = x'^k \vec{e}\,'_k = x'^k A_k^i \vec{e}_i \ . \tag{8}$$

Tensor calculus

As a result of the uniqueness of the decomposition of a vector on a basis, we can equalize the coefficients of the vectors $\vec{e}'_k$ and $\vec{e}_i$ we obtain

$$\boxed{x'^k = A'^k_i x^i \ ,} \tag{9}$$

and

$$\boxed{x^i = A^i_k x'^k \ .} \tag{10}$$

Tensor calculus

C. Contravariant and covariant components

For a Euclidean vector space E_n, related to any basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3, .., \vec{e}_n)$ the scalar product $\vec{x} = x^i \vec{e}_i$ by a basis vector $\vec{e}_j$ reads

$$\vec{x}.\vec{e}_j = (x^i \vec{e}_i) \cdot \vec{e}_j = x^i (\vec{e}_i \cdot \vec{e}_j) = x^i g_{ij} \equiv x_j \ . \tag{11}$$

These scalar products, denoted x_j are called the covariant components, in the basis $(\vec{e}_i)$, of the vector $\vec{x}$. These components are therefore defined by

$$\boxed{x_j \equiv \vec{x} \cdot \vec{e}_j \ .} \tag{12}$$

They will be noted using lower indices. We will see later that these components are naturally introduced for certain physics vectors, such as the gradient vector. On the other hand, the notion of covariant component is essential for tensors.

$$\vec{\nabla} \equiv \partial_i$$

→ Important issue of the « covariant derivative »

Tensor calculus

$$\vec{x}.\vec{e}_j = (x^i\vec{e}_i)\cdot\vec{e}_j = x^i(\vec{e}_i\cdot\vec{e}_j) = x^i g_{ij} \equiv x_j \ . \tag{11}$$

The relationship (11) shows that the covariant components x_j are related to the classical components of the vector $\vec{x}$. To distinguish them, the latter are called the contravariant components of the $\vec{x}$ vector. The contravariant components are therefore numbers x^i such that

$$\boxed{\vec{x} = x^i\vec{e}_i \ .} \tag{13}$$

They will be noted using upper indices. The study of basis changes will justify the naming of the various components.

Tensor calculus

Any basis

As an example, let's consider two vectors $\vec{e}_1$ and $\vec{e}_2$ of classical geometry with arbitrary directions and lengths. We have (see figure) $\vec{A} \equiv O\vec{M}$, $O\vec{M}' = x^1\vec{e}_1$, $O\vec{M}'' = x^2\vec{e}_2$, and $\vec{A} = x^1\vec{e}_1 + x^2\vec{e}_2$. The numbers are the contravariant components of vector $\vec{A}$. Let's use the classic scalar product expression to express the covariant components. It results

$$x_1 = \vec{A} \cdot \vec{e}_1 = \|\vec{A}\| \|\vec{e}_1\| \cos\alpha \ , \tag{14}$$

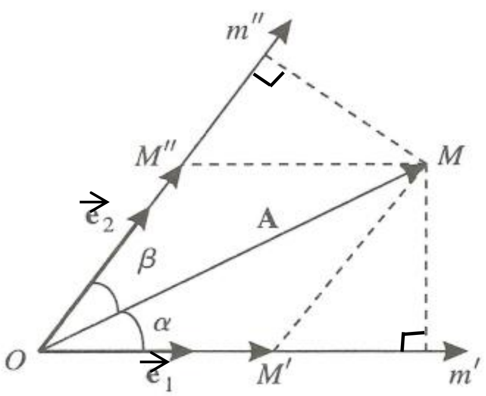
and

$$x_2 = \vec{A} \cdot \vec{e}_2 = \|\vec{A}\| \|\vec{e}_2\| \cos\beta \ . \tag{15}$$

If the basis vectors $\vec{e}_1$ and $\vec{e}_2$ have norms equal to unity, then the orthogonal projections m' and m'' of point M represent the covariant components of $\vec{A}$.

Note: In an orthonormal basis, the covariant and contravariant components are identical since

$$x_j = \vec{x} \cdot \vec{e}_j = (x^i\vec{e}_i) \cdot \vec{e}_j = x^i(\vec{e}_i \cdot \vec{e}_j) = x^i\delta_{ij} = x^j \ . \tag{16}$$



Tensor calculus

D. Change of basis

Consider two distinct basis $(\vec{e}_i)$ and $(\vec{e}'_i)$ of a Euclidean vector space $\mathcal{E}_n$, linked by the relations

$$\vec{e}_i = A'^k_i \vec{e}'_k \ , \tag{17}$$

and

$$\vec{e}'_k = A^i_k \vec{e}_i \ . \tag{18}$$

Let be x^i and x'^k the contravariant components of a vector $\vec{x}$ respectively in the basis $(\vec{e}_i)$ and $(\vec{e}'_i)$. We have seen that

$$x^i = A^i_k x'^k \ , \tag{19}$$

and

$$x'^k = A'^k_i x^i \ . \tag{20}$$

Therefore the transformation relations of the contravariant components (19,20) are the opposite of those of the basis vectors (17,18), with the quantities A and A' exchanging, hence the name of these components.

Tensor calculus

E. Scalar product

From $\vec{x} = x^i \vec{e}_i$ and $\vec{y} = y^i \vec{e}_i$ we obtain

$$\vec{x} \cdot \vec{y} = \left(x^i \vec{e}_i\right) \cdot \left(y^j \vec{e}_j\right) = x^i y^j \left(\vec{e}_i \cdot \vec{e}_j\right) = g_{ij} x^i x^j \ . \tag{21}$$

From

$$\vec{x} \cdot \vec{y} = g_{ij} x^i y^j = \vec{y} \cdot \vec{x} = g_{ji} y^j x^i \ , \tag{22}$$

we get

$$\boxed{g_{ij} = g_{ji} \ .} \tag{23}$$

F. norm of a vector

We have

$$\text{norme of } \vec{x} = (\vec{x} \cdot \vec{x})^{1/2} = (g^{ji} x_j x_i)^{1/2} \ . \tag{24}$$

A vector's norm is also its length. In an orthonormal base, the contravariant and covariant components are identical. We have the important relation

$$\boxed{x^j = g^{ji} x_i \ .} \tag{25}$$

Tensor calculus

G. Reciprocal basis

Let be any basis $(\vec{e}_i)$ of an Euclidean vector space. By definition, n vectors $\vec{e}^k$ which satisfy the following relations:

$$\boxed{\vec{e}_i \cdot \vec{e}^k = \delta_{ik} \ ,} \tag{26}$$

are called the reciprocal vectors of the $\vec{e}^i$ vectors. They will be noted with higher indices. Each reciprocal vector $\vec{e}^k$ is orthogonal to all vectors $\vec{e}_i$, except for $k = i$.

Example - Let three vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ form a basis for the vectors of classical geometry. Note $v = \vec{e}_1 \cdot (\vec{e}_2 \wedge \vec{e}_3)$, where the symbol $\wedge$ represents the vector product. The following vectors:

Dual space!

$$\left\{ \begin{aligned} \vec{e}^1 &= \frac{\vec{e}_2 \wedge \vec{e}_3}{v} && (27) \\ \vec{e}^2 &= \frac{\vec{e}_3 \wedge \vec{e}_1}{v} && (28) \\ \vec{e}^3 &= \frac{\vec{e}_1 \wedge \vec{e}_2}{v} \ , && (29) \end{aligned} \right.$$

verify relations (26) and constitute the reciprocal system of vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. In crystallography, these are the vectors of the associated Fourier space.

Tensor calculus

We have the following relations (without demonstration):

$$\boxed{\begin{aligned} \vec{e}_i &= g_{ik}\vec{e}^k \,, && (30) \\ \vec{x} &= x_i\vec{e}^i \,, && (31) \\ \vec{x}\cdot\vec{e}^k &= x^k \,, && (32) \\ g^{ik} &= \vec{e}^i\cdot\vec{e}^k \,, && (33) \end{aligned}}$$

and

$$g_{ik}g^{kj} = \delta_i^j \,. \tag{34}$$

In conclusion, reciprocal bases play strictly symmetrical roles, with components that are contravariant in one basis becoming covariant in the reciprocal basis and vice versa.

Tensor calculus

II. EXAMPLES OF EUCLIDEAN TENSORS

A. Covariant components of the fundamental tensor

In the previous chapter, we used the quantities g_{ij}, defined from the scalar product of the basis vectors $\vec{e}_i$ of a Euclidean vector space E_n by

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j \ . \tag{35}$$

These n^2 quantities constitute the covariant components of a tensor called the **fundamental tensor** or **metric tensor**.

Tensor components are generally classified in the form of an ordered ordered table

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{36}$$

Example - Consider the basis vectors of the E_3 vector space $\vec{e}_1 = (2,0,0), \vec{e}_2 = (0,1,0), \vec{e}_3 = (0,0,3)$. We have:

$$g_{11} = 4, g_{22} = 1, g_{33} = 9, g_{ij} = 0 \text{ if } i \neq j \ , \tag{37}$$

leading to

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix} \tag{38}$$

Tensor calculus

Change of basis - Let's study how the quantities g_{ij} vary when we perform a change of the basis. Let $\vec{e}'_k$ be another basis linked to the previous one by:

$$\vec{e}_i = A_i'^k \vec{e}'_k \;;\; \vec{e}'_k = A_k^i \vec{e}_i \;. \tag{39}$$

We have

$$g_{ij} = \left(A_i'^k \vec{e}'_k\right) \cdot \left(A_j'^l \vec{e}'_l\right) = \left(A_i'^k A_j'^l\right) (\vec{e}'_k \cdot \vec{e}'_l) \;. \tag{40}$$

By using $g'_{kl} = \vec{e}'_k \cdot \vec{e}'_l$ it leads to

$$g_{ij} = A_i'^k A_j'^l g'_{kl} \;, \tag{41}$$

and

$$g'_{ij} = A_i^k A_j^l g_{kl} \;. \tag{42}$$

The covariant components g_{ij} of the fundamental tensor are no longer transformed like the covariant components of a vector but by involving the product of the quantities $A_i'^k$.

Tensor calculus

B. Basis change properties

By definition $t_{ij} = A'^k_i A'^l_j t'_{kl}$ and $t'_{kl} = A^i_k A^j_l t_{ij}$ are, by definition, the covariant components of a second-order tensor.

On can show that

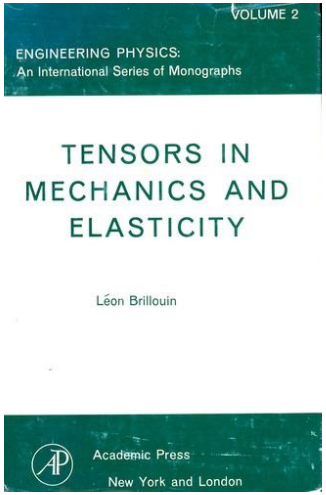
$$\boxed{\begin{aligned} u'^l_k &= A^i_k A'^l_j u^j_i \ , && (43) \\ v^{ij} &= A^i_k A^j_l v'^{kl} \ , && (44) \\ &\text{and} \\ v'^{kl} &= A'^k_i A'^l_j v^{ij} \ . && (45) \end{aligned}}$$

Note that u^j_i is called a second-order mixed tensor.

Tensor calculus

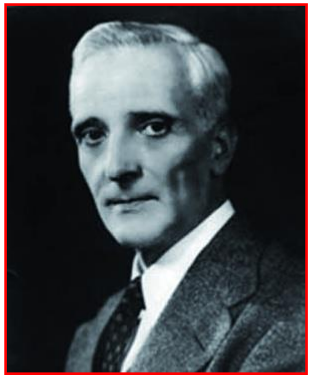
Examples of second-order tensors

- Metric tensor g_{ij}.
- Tensor of inertia I_{ij}.
- The conductivity tensor σ_{ij}. We have $j^i = \sigma^i_k E^k$.
- Tensorial product of two vectors: $u^{ij} = x^i y^j$, $u^j_i = x_i y^j$, $u_{ij} = x_i y_j$.
- Electromagnetic field tensor $F_{\mu\nu}$.
- Stress-energy tensor, sometimes called the stress-energy-momentum tensor or the energy-momentum tensor, $T^{\mu\nu}$.
- ...



Examples of third-order tensors

- $u^{ijk} = x^i y^j z^k$ and all the combination of indices (4 different types).
- Nonlinear optical susceptibilities are tensors. e.g. $P_k = \chi_{ijk} E^i E^j$.
- *Tensors in mechanics and elasticity* by Leon Brillouin.
- ...



(1889-1969)

Tensor calculus

By using the general definition we have the following relations

- $u^{ijk} = A^i_l A^j_m A^k_p u'^{lmp}$,
- $u_{ijk} = A'^l_i A'^m_j A'^p_k u'_{lmp}$,
- $u^i_{jk} = A^i_l A'^m_j A'^p_k u'^l_{mp}$,
- $u^{ij}_k = A^i_l A^j_m A'^p_k u'^{lm}_p$.

Tensor calculus

III. TENSORIAL ALGEBRA

Tensor product - In mathematics, the "tensor product" $E_p \otimes E_n$ of two vector spaces E_p and E_n is a vector space to which is associated a bilinear map $E_p \times E_n \to E_p \otimes E_n$ that maps a pair $(\vec{v}, \vec{w})$ $\vec{v} \in E_p$, $\vec{w} \in E_n$ to an element of $E_p \otimes E_n$ denoted $\vec{v} \otimes \vec{w}$.

Tensor product of identical spaces - In practice, we often have to use tensors formed from vectors belonging to identical E_n vector spaces.

Tensor product of p vectors - In general, we can form the space $E_n^{(p)}$ corresponding to p times the tensor multiplication of the space E_n by itself, i.e.:

$$E_n^{(p)} = E_n \otimes E_n \otimes ... \otimes E_n \ . \tag{46}$$

If we now denote p vectors of E_n as $\vec{x}_1 = x^{i_1}\vec{e}_{i_1}, \vec{x}_2 = x^{i_2}\vec{e}_{i_2}, ..., \vec{x}_p = x^{i_p}\vec{e}_{i_p}$ the tensor products of $E_n^{(p)}$ are tensors of order p of the form

$$\vec{x}_1 \otimes \vec{x}_2 \otimes ... \otimes \vec{x}_p = x^{i_1}x^{i_2}...x^{i_p}(\vec{e}_{i_1} \otimes \vec{e}_{i_2} \otimes ... \otimes \vec{e}_{i_p}) \ , \tag{47}$$

with $i_1, i_2, ..., i_p = 1$ to n.

Tensor calculus

A. Tensor classification

1. *Order 0*

Scalars are conveniently called zero order tensors. To form a scalar, we use the scalar product of tensors of the same rank. The simplest example is the scalar product of two vectors (rank one tensor).

$$\vec{x} \cdot \vec{y} = g_{ij} x^i x^j \ . \tag{48}$$

Note: There are no indices left!

We can generalize to tensors of higher rank e.g.

$$u^{jk} v^{lm} g_{jl} g_{km} = u^{jk} g_{jl} v^{lm} g_{km} = u_l^k v_k^l \ . \tag{49}$$

Note: Once again there are no indices left!

2. *Order 1*

Vectors x^i, $u^{jk} v_k$, $P^k = \chi_{ij}^k E^i E^j$...only one indices left.

Tensor calculus

3. Order 2

g_{ij} etc...

Rule: By multiplying by a quantity g_{ij} or g^{ij} and summing, we can place each of the indices of a tensor in a position that is either contravariant or covariant. We have:

$$\left\{\begin{aligned} u^{i_1 i_2 \ldots i_p} &= g^{i_1 k_1} g^{i_2 k_2} \ldots g^{i_p k_p} u_{k_1 k_2 \ldots k_p} && (50)\\ u_{i_1}^{\;i_2 i_3 \ldots i_p} &= g_{i_1 k_1} u^{k_1 i_2 \ldots i_p} && (51)\\ u_{i_1 i_2}^{\;\;i_3 \ldots i_p} &= g_{i_1 k_1} g_{i_2 k_2} u^{k_1 k_2 i_3 \ldots i_p} && (52)\end{aligned}\right.$$

Theorem: for a sequence of n^3 quantities, related to a basis of a tensor space $E_n^{(3)}$, to be considered as components of a tensor, it is necessary and sufficient for these quantities to be linked together, in two different bases of $E_n^{(3)}$, by the previous formulae for transforming the components. This conclusion can be generalised to n^p quantities that can constitute the components of a tensor of a tensor space $E_n^{(p)}$.

B. Operations on tensors

1. Addition of tensors of the same order

In order to add up, the tensors must obviously be related to the same basis. The sum of the covariant components of two tensors gives the covariant components of their sum. The same applies to the mixed components relative to the same basis. We have

$$\mathbb{T} = \mathbb{U} + \mathbb{V} \,, \tag{53}$$

and

$$(u^{ijk} + v^{ijk})\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}_3 = t^{ijk}\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}_3 \,. \tag{54}$$

2. Tensorial multiplication of tensors

$$w^{ijklm}\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}_3 \otimes \vec{e}_4 \otimes \vec{e}_5 = u^{ij}\vec{e}_1 \otimes \vec{e}_2 \times v^{klm}\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}_3 \,, \tag{55}$$

leading to

$$\mathbb{W} = \mathbb{U} \otimes \mathbb{V} \,, \tag{56}$$

and for the components

$$w^{ijklm} = u^{ij}v^{klm} \,. \tag{57}$$

Tensor calculus

3. *Contraction of indices*

In addition to the operations of addition and tensor multiplication, there is an operation that can be used to obtain other tensors from a given tensor. This is the index contraction operation.

Example 1: scalar product - Consider the tensor product of two vectors $\vec{x}$ and $\vec{y}$ with respective contravariant x^i and covariant y_j components. The components of the tensor product $\mathbb{V}$ of these two vectors are:

$$v^i_j = x^i y_j \ . \tag{58}$$

It's a mixed second-order tensor. Let's add up the different components of the $\mathbb{V}$ tensor such that $i = j$, it leads to:

$$v = x^i y_i \ . \tag{59}$$

The quantity v is a zero-order tensor or scalar. Such an addition of indices of different variance is, by definition, the operation of contraction of the indices of the tensor $\mathbb{V}$. This operation has enabled us to go from a second-order tensor to a zero order tensor; the $\mathbb{V}$ tensor has been stripped of one covariance and one contravariance.

Example 2: third-order tensor - Let's take the example of a tensor $\mathbb{U}$ whose mixed components are u^{ij}_k. Let's consider some of its components such that $j = k$, i.e. the quantities u^{ij}_j and add them together, we obtain

$$v^i = u^{i1}_1 + u^{i2}_2 + ... + u^{in}_n = \delta^k_j u^{ij}_k \ . \tag{60}$$

Tensor calculus

These new quantities v^i form the components of a first-order tensor $\mathbb{V}$ (vector), as we shall see. The v^i quantities are contracted components of the $\mathbb{U}$ tensor.

Tensor of any order - The contraction operation therefore consists in choosing two indices, one covariant, the other contravariant, equating them and summing with respect to this twice-repeated index.

In general, the contraction of a tensor makes it possible to form a tensor of order $(p-2)$ from a tensor of order p. Of course, the contraction operation can be repeated. For example, a tensor of even order, $2p$, will become a scalar after p contractions and an odd-order tensor, $2p+1$, will become a vector.

Tensor calculus

4. *Tensoriality criteria*

One way of recognizing the tensor character of a sequence of quantities is to study the way these quantities are transformed during a change of basis and to check the conformity of the transformation formulas.

The contracted multiplication will enable us to obtain another tensoriality criterion, which may be easier and quicker to use than the previous one.

The demonstrations will be carried out on examples, but can be generalized to tensors of any order.

Completely contracted product - Let's consider the sequence of n^3 quantities u_k^{ij}, attached to a basis $\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}^k$ and look for a way to determine whether they can constitute the components of a tensor. Let, on the other hand, be vectors $\vec{x} = x_i \vec{e}^i, \vec{y} = y_j \vec{e}^j, \vec{z} = z^k \vec{e}_k$. If the sequence u_k^{ij} is tensorial, then the contracted product

$$\alpha = u_k^{ij} x_i y_j z^k \ , \tag{61}$$

is a scalar quantity, invariant to base changes, according to the properties of the contracted product.

By generalizing, we arrive at the following conclusion: for a set of n^{p+q} quantities, with p upper indices and q lower indices, to be tensorial, it is necessary and sufficient that their product, completely contracted by the contravariant components of any p vectors and the covariant components of any q vectors, is a quantity that remains invariant to the change of basis.

Tensor calculus

5. *Special tensors*

- Symmetrical: $u_{ij} = u_{ji}$; example: the fundamental tensor $g_{ij} = \vec{e}_i \cdot \vec{e}_j$.
- Partially symmetrical: $u_l^{ijk} = u_l^{jik}$
- Antisymmetric tensor: $u^{ij} = -u^{ji}$
- partially antisymmetrical: $u_l^{ijk} = -u_l^{jik}$

Note 1 (important: beginning of the notion of covariance): Let's use the general criterion of tensoriality to demonstrate the tensorial character of g_{ij}. The expression $g_{ij}x^i y^j$ is a completely contracted product of the contravariant components x^i and y^j of an arbitrary first-order tensor (a vector here). Since the scalar product is an invariant quantity with respect to basis changes, it follows that the n^2 quantities g_{ij} are the covariant components of a tensor.

Note 2 : $a_{ij}x^i y^j = 1 \to a'_{ij}x'^i y'^j = 1$ if $a'_{km} = A^i_k A^j_m a_{ij}$.

Note 3: a tensor will be completely antisymmetrical if any index transposition of the same variance changes the corresponding component into its opposite.

Note 4: completely symmetrical third-order tensor: $u_l^{ijk} = u_l^{jik} = u_l^{kji} = u_l^{ikj}$.

Note 5: Any tensor u_{ij} can be expressed as the sum of a symmetrical tensor and an antisymmetric tensor. We have:

$$\boxed{u^{ij} = \frac{1}{2}(u^{ij} + u^{ji}) + \frac{1}{2}(u^{ij} - u^{ji})\ .} \tag{62}$$

The first term of the above sum is a symmetrical tensor and the second, an antisymmetrical tensor.

Tensor calculus

6. Exterior product of two vectors

Given two vectors $\vec{x} = x^i\vec{e}_i$ and $\vec{y} = y^j\vec{e}_j$ of a vector space E_n, let's form the following antisymmetric quantities:

$$u^{ij} = x^i y^j - x^j y^i \ . \tag{63}$$

These are the components of an antisymmetric tensor $\mathbb{U}$, denoted $\vec{x} \wedge \vec{y}$, whose decomposition on the basis $\vec{e}_i \otimes \vec{e}_j$ is written as

$$\mathbb{U} = \vec{x} \wedge \vec{y} = (x^i y^j - x^j y^i)\vec{e}_i \otimes \vec{e}_j \ . \tag{64}$$

The second-order tensor $\vec{x} \wedge \vec{y}$ is called the exterior product of the vectors $\vec{x}$ and $\vec{y}$; it is also said to be a bivector.

Strict components of the exterior product - Of the n^2 components of the exterior product, n components are zero and the other $n(n-1)$ components have opposite values. We can therefore consider that half of the latter components is sufficient to characterize the tensor, and we'll say that the exterior product has $n(n-1)/2$ strict components.

Tensor calculus

Note that for $n = 3$, the number of strict components of the exterior product of two vectors is also equal to 3. This makes it possible to form, with the strict components of the bivector $\vec{x} \wedge \vec{y}$, the components of the vector product $\vec{z} = \vec{x} \wedge \vec{y}$.

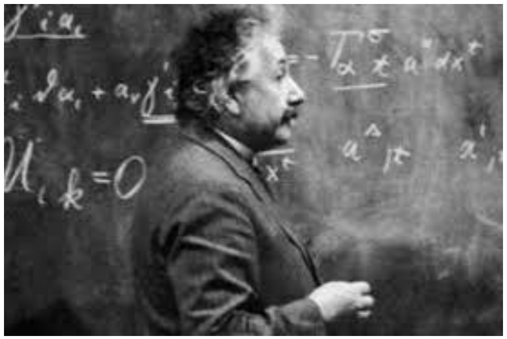
To do this, we set

$$\left\{ \begin{aligned} u^{23} &= x^2 y^3 - x^3 y^2 = z^1 && (65) \\ u^{31} &= x^3 y^1 - x^1 y^3 = z^2 && (66) \\ u^{12} &= x^1 y^2 - x^2 y^1 = z^3 \, . && (67) \end{aligned} \right.$$

A vector product therefore only exists for three-dimensional spaces and and we know that it only transforms like a vector for certain basis changes. It's an axial vector. The $\vec{z}$ vector is said to be the adjoint tensor of the $\mathbb{U}$ tensor. This is a special example of the adjoint tensor of an antisymmetric tensor.

General Relativity (GR)

M1 - Physique 2023-2024



Paul-Antoine Hervieux
Unistra/IPCMS
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$$g_{\mu\nu}$$

AE+GR (1907-1917)

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

II) preliminaries of physics and mathematics

- General considerations (Einstein equivalence principle…)
- Field theory
- Tensors
- Special relativity, and complements (covariant formulation of *em*)

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

4 formulations of the Maxwell equations

Integral formulation

$$\oiint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, dV$$

$$\oiint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0$$

$$\oint_{\partial\Sigma} \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \vec{B} \cdot \mathrm{d}\vec{S}$$

$$\oint_{\partial\Sigma} \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \left(\iint_{\Sigma} \vec{j} \cdot \mathrm{d}\vec{S} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \vec{E} \cdot \mathrm{d}\vec{S} \right)$$

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Historically, this is the first formulation (~1850)

4 formulations of the Maxwell equations

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$\longrightarrow$

Differential formulation

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\overrightarrow{\nabla} . \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} . \overrightarrow{E} = \frac{\rho}{\varepsilon_0}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{B} = \mu_0 \overrightarrow{j} + \varepsilon_0\mu_0\frac{\partial \overrightarrow{E}}{\partial t}$$

From Maxwell and Heaviside (~1880)

4 formulations of the Maxwell equations

Integral formulation

$$\oiint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_{\Omega} \rho\, dV$$

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$$\overrightarrow{\nabla} \wedge \overrightarrow{B} = \mu_0 \overrightarrow{j} + \varepsilon_0\mu_0\frac{\partial \overrightarrow{E}}{\partial t}$$

$\longrightarrow$

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha\left(\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0$$

Tensorial (or covariant) formulation

4 formulations of the Maxwell equations

Integral formulation

$$\oiint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_{\Omega} \rho\, dV$$

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$\longrightarrow$

Differential formulation

$$\vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\wedge\vec{B} = \mu_0\,\vec{j} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

$\longrightarrow$

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha\left(\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0$$

As the equations are written in terms of tensors, we can immediately see that the form is conserved in a Lorentz-Poincaré transformation. This is why we speak of covariant notations.

Tensorial (or covariant) formulation

4 formulations of the Maxwell equations

Integral formulation

$$\oiint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_{\Omega} \rho \, dV$$

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$\longrightarrow$

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha \left(\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \right) = 0$$

Tensorial (or covariant) formulation

$\longrightarrow$

$$\mathrm{d}F = 0$$

$$\mathrm{d} * F = \mu_0 * J$$

Formulation in terms of differential forms

4 formulations of the Maxwell equations

Integral formulation

$$\oiint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_{\Omega} \rho\, dV$$

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$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha\left(\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0$$

Tensorial (or covariant) formulation

$$\mathrm{d}F = 0$$

$$\mathrm{d} * F = \mu_0 * J$$

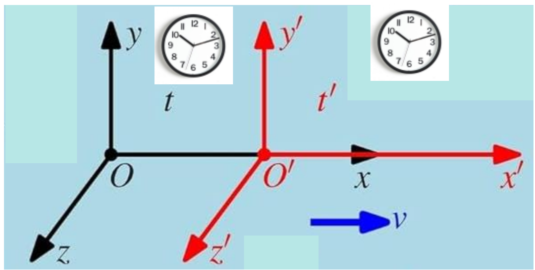
Formulation in terms of differential forms

Notations free of any frame of reference, of any coordinates, of any metric!

Covariant formulation of classical electromagnetism

The Lorentz-Poincaré transformation

The most common form of the Lorentz transformation is:



$$\begin{cases} t' = \gamma\left(t - \dfrac{vx}{c^2}\right) & (1) \\ x' = \gamma\left(x - vt\right) & (2) \\ y' = y & (3) \\ z' = z & (4) \end{cases}$$

where (t, x, y, z) and (t', x', y', z') represent the coordinates of an event (space-time) in two inertial reference frames whose relative velocity v is parallel to the x axis, c is the speed of light, and the Lorentz factor which is given by: $\gamma = \frac{c}{\sqrt{c^2-v^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} = \frac{dt}{d\tau}$ where v is the relative velocity between inertial reference frames, c is the speed of light in vacuum, β is the ratio of v to c,t is coordinate time, and τ is the proper time for an observer (measuring time intervals in the observer's own frame).

- *Maxwell's equations are invariant under the Lorentz-Poincaré transformation*

Covariant formulation of classical electromagnetism

In matrix form, the Lorentz transformations can be written as:

same physical dimensions!

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{5}$$

Orthogonal transf.

where the matrix Λ satisfies the following expected properties: (i) $det(\Lambda) = +1$, which means that the transformation preserves the orientation of the space; (ii) $\Lambda^T \, \eta \, \Lambda = \eta$, where η is the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$, which means that the matrix is pseudo-orthogonal and preserves the space-time interval of Minkowski space.

The matrix representation of the Minkowski metric tensor is: $\boxed{\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$.

This writing in 4×4 matrix form corresponds to the standard representation of the Lorentz group. Objects that transform under this representation are quadrivectors (here, the time-position quadrivector).

In special relativity, a four-vector (or 4-vector) is an object with four components, which transform in a specific way under Lorentz transformations. We have

$$\boxed{x'^{\nu} = \Lambda'^{\nu}_{\ \mu} x^{\mu}} \tag{6}$$

Covariant formulation of classical electromagnetism

and for a general tensor

$$\boxed{T'^{\alpha'\beta'\cdots\zeta'}_{\theta'\iota'\cdots\kappa'} = \Lambda'^{\alpha'}{}_{\mu}\Lambda'^{\beta'}{}_{\nu}\cdots\Lambda'^{\zeta'}{}_{\rho}\Lambda_{\theta'}{}^{\sigma}\Lambda_{\iota'}{}^{\upsilon}\cdots\Lambda_{\kappa'}{}^{\zeta}T^{\mu\nu\cdots\rho}_{\sigma\upsilon\cdots\zeta}\,.} \tag{7}$$

The most important quadrivectors are:

- Displacement (vector): $x^\alpha = (ct, \vec{x}) = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$.
- Four-velocity: $u^\alpha = \gamma(c, \vec{u})$ where γ is the Lorentz factor at the 3-velocity $\vec{u}$.
- Four-momentum: $p^\alpha = (E/c, \vec{p}) = m_0 u^\alpha$ where $\vec{p}$ is the 3-momentum, E is the total energy, and m_0 is the rest mass.
- Four-gradient: $\partial^\nu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$; we have $\partial_\nu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$. One can use $\partial^\nu = g^{\mu\nu}\partial_\mu$ where $g^{\mu\nu} = \eta^{\mu\nu}$ is the Minkowski metric tensor.
- d'Alembertian operator: $\partial^2 \equiv \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t} - \Delta$.
- Four-current: $J^\alpha = (c\rho, \vec{j})$ where ρ is the electric charge density and $\vec{j}$ the electric current density.
- Four-potential: $A^\alpha = (\phi/c, \vec{A})$ where ϕ is the electric scalar potential and $\vec{A}$ the vector potential).

Covariant formulation of classical electromagnetism

The electromagnetic tensor

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic tensor is the combination of the electric and magnetic fields into a covariant antisymmetric tensor.

By using the usual relations $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \wedge \vec{A}$ we have:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{8}$$

We have also $F^{\mu\nu} = \eta^{\mu\alpha} F_{\alpha\beta} \eta^{\beta\nu}$ leading to

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{9}$$

Since $F^{\mu\nu}$ is a rank 2 tensor, it transforms as follows

$$\boxed{F'^{\mu'\nu'} = \Lambda'^{\mu'}{}_{\mu} \Lambda'^{\nu'}{}_{\nu} F^{\mu\nu} \ .} \tag{10}$$

Covariant formulation of classical electromagnetism

Maxwell equations

The two inhomogeneous Maxwell's equations, Poisson (ρ) and Ampère ($\vec{j}$) combine into:

$$\boxed{\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \ .} \tag{11}$$

The two homogeneous Maxwell's equations, Faraday's law of induction and Gauss's law for magnetism combine to form:

$$\boxed{\partial_\alpha \left(\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \right) = 0} \tag{12}$$

$$F^{*\mu\nu} = \widetilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Dual tensor

$$\boxed{\partial_\mu \widetilde{F}^{\mu\nu} = 0}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol.

In the four-dimensional space-time, the Levi-Civita symbol (not a tensor) is defined by:

$$\varepsilon_{ijkl} = \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \ . \tag{13}$$

Some examples:

$$\varepsilon_{0321} = -\varepsilon_{0123} = -1 \tag{14}$$

$$\varepsilon_{1023} = -\varepsilon_{0123} = -1 \tag{15}$$

Covariant formulation of classical electromagnetism

Lorentz's gauge

Lorentz's gauge: $\partial_\alpha A^\alpha = 0$ or more familiar $\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}$. In the Lorenz gauge, the microscopic Maxwell's equations can be written as: $\partial_\beta\partial^\beta A^\alpha = \mu_0 J^\alpha$.

Lorentz's gauge

$$\boxed{\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0} \longrightarrow \begin{cases} \Box\vec{A}(\vec{r},t) = \dfrac{1}{\epsilon_0 c^2}\vec{j}(\vec{r},t) \\ \Box\phi(\vec{r},t) = \dfrac{1}{\epsilon_0}\rho(\vec{r},t)\,. \end{cases}$$

Wave equations

- Continuity equation: $\boxed{J^\beta{}_{,\beta} \equiv \partial_\beta J^\beta = 0}$ expresses charge conservation. Using the definition of quadrivectors, we check that we find the usual relationship $\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$.

- It is associated with the gauge invariance of Maxwell's equations (Noether's theorem for internal symmetry).

Covariant formulation of classical electromagnetism

Maxwell's equations in terms of differential forms

In general, an k-differential form (or simply k-form) in variables x^1, ..., x^n will be an expression

$$\boxed{\omega = \frac{1}{k!} t_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \ ,} \tag{16}$$

where the coefficients $t_{i_1,\ldots,i_k}$ (coefficients of the tensor $\mathbb{T}$) are smooth functions of the variables x^1, ..., x^n and antisymmetric in the indices. Remember that $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$.

Examples:

1. 0-form: a function $\omega = f(x^1, \ldots, x^n)$.

2. 1-form: $\omega = A_\alpha \mathrm{d}x^\alpha$

Each k-form ω is associated with an $(k+1)$-form $\mathrm{d}\omega$ called the exterior derivative of ω which is written as

$$\boxed{\mathrm{d}\omega = \sum_{i_1<\ldots<i_k} \sum_{i_0} \frac{\partial t_{i_1\ldots i_k}}{\partial x^{i_0}} \mathrm{d}x^{i_0} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \ .} \tag{17}$$

Example: for a 0-form (i.e. a function $\omega = f(x^1, \ldots, x^n)$), we find the expression of the 1-form (differential of a function):

$$\mathrm{d}\omega \equiv \mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \mathrm{d}x^j = \partial_j f \mathrm{d}x^j \ . \tag{18}$$

Covariant formulation of classical electromagnetism

Maxwell's equations in terms of differential forms

From this definition follows the general Stockes' formula

$$\boxed{\int_{\partial\Sigma} \omega = \int_{\Sigma} \mathrm{d}\omega \ .} \tag{19}$$

Here Σ is an $(k+1)$ dimensional oriented variety and $\partial\Sigma$ is its boundary.

Examples:

1. $\int_a^b \mathrm{d}f = \int_a^b f'(x)dx = f(b) - f(a)$.

$f = \omega$

2. Divergence theorem: $\int_V (\vec{\nabla} \cdot \vec{F})dV = \int_{\partial V} \vec{F} \cdot d\vec{S}$.

One of the basic relations is Poincaré's Lemma:

$$\boxed{\mathrm{d}(\mathrm{d}\omega) = 0 \ .} \tag{20}$$

Example: $\mathrm{d}(\mathrm{d}A) = \mathrm{d}F = 0$ with $A = A_\alpha \mathrm{d}x^\alpha$. A_α is the quadrivector potential. A form ω is said to be closed if $\mathrm{d}\omega = 0$.

Covariant formulation of classical electromagnetism

Maxwell's equations in terms of differential forms

All electromagnetism allows itself to be summarized in the language of 2-forms. Using the formalism of differential forms, Maxwell's equations can be written in an extremely simple way

$$\boxed{\begin{aligned} \mathrm{d}F &= 0 \\ \mathrm{d} * F &= \mu_0 * J \ , \end{aligned}} \qquad \begin{aligned} (21) \\ (22) \end{aligned}$$

where $F = \mathrm{d}A$ with A a 1-form $A = A_\alpha \mathrm{d}x^\alpha$ and $J = J_\alpha \mathrm{d}x^\alpha$. $*F$ is the dual electromagnetic tensor.

$$(*\omega)_{i_{p+1}\ldots i_n} = \frac{1}{p!}\eta_{i_1\ldots i_n}\omega^{i_1\ldots i_p}$$

$$\eta_{i_1\ldots i_n} = \sqrt{|g|}\varepsilon_{i_1\ldots i_n} \qquad |g| := \left|\det(g_{ij})\right|$$

- Operation *: Hodge dual or Hodge star operation
- **F* is also called the adjoint tensor of *F*

F 2-form → **F* 2-form

$$*F = \frac{1}{2}(*F)_{\alpha\beta}dx^\alpha \wedge dx^\beta$$

$$(*F)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$$

J 1-form → **J* 3-form

$$*J = (*J)_{\alpha\beta\gamma}dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

$$(*J)_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta}J^\delta$$

Covariant formulation of classical electromagnetism

Maxwell's equations as a field theory

The total action of the "particles+field" system is as follows:

$$S = S_{part} + S_{field} + S_{int} \ , \tag{31}$$

with

$$S_{part} = -mc \int ds \ , \tag{32}$$

$$S_{field} = -\frac{1}{4\mu_0}\frac{1}{c}\int F^{\alpha\beta}F_{\alpha\beta}d\Omega \ , \tag{33}$$

and

$$S_{int} = -\frac{1}{c}\int A_\alpha J^\alpha d\Omega \ , \tag{34}$$

with $d\Omega = cdtdxdydz = cdtdV$.

Expression in terms of differential forms

$$S_{field} = \int -\frac{1}{2\mu_0} F \wedge *F$$

$$S_{int} = \int A \wedge *J$$

Coordinate system independent!
no volume term!

Covariant formulation of classical electromagnetism

Maxwell's equations as a field theory

The lagrangian density of the "particles+field" system is:

$$\boxed{\begin{aligned} \mathcal{L} &= \mathcal{L}_{field} + \mathcal{L}_{int} \\ &= -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_\alpha J^\alpha \ , \end{aligned}} \qquad \begin{aligned} (35) \\ (36) \end{aligned}$$

and

$$\boxed{L_{part} = -mc^2 \sqrt{1-\beta^2} \ .} \qquad (37)$$

By using $S_{part} \equiv \int L_{part} dt$ with $L_{part} = -mc^2\sqrt{1-\beta^2}$ and knowing that $\Delta\tau \equiv t'_2 - t'_1 = \int_{t_1}^{t_2} dt \sqrt{1-\beta^2} = \frac{1}{c}\int ds$ we get $S_{part} = -mc\int ds$. Moreover, if $v \ll c$, $L_{part} = \frac{1}{2}mv^2$ the traditional non-relativistic expression.

Covariant formulation of classical electromagnetism

Maxwell's equations as a field theory: field equations

The Euler-Lagrange equation for the electromagnetic lagrangian density $\mathcal{L}\left(A_\alpha, \partial_\beta A_\alpha\right)$ can be stated as follows (variation with respect to A_α, only $\mathcal{L}_{field}$ and $\mathcal{L}_{int}$ are concerned): $\partial_\beta\left[\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}\right] - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0$. By noting

$$F^{\lambda\sigma} = F_{\mu\nu}\eta^{\mu\lambda}\eta^{\nu\sigma} \tag{38}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{39}$$

$$\frac{\partial\left(\partial_\mu A_\nu\right)}{\partial\left(\partial_\rho A_\sigma\right)} = \delta^\rho_\mu \delta^\sigma_\nu \ , \tag{40}$$

the expression inside the square bracket is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= -\frac{1}{4\mu_0}\,\frac{\partial\left(F_{\mu\nu}\eta^{\mu\lambda}\eta^{\nu\sigma}F_{\lambda\sigma}\right)}{\partial\left(\partial_\beta A_\alpha\right)} & (41)\\ &= -\frac{1}{4\mu_0}\,\eta^{\mu\lambda}\eta^{\nu\sigma}\left(F_{\lambda\sigma}\left(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu\right) + F_{\mu\nu}\left(\delta^\beta_\lambda\delta^\alpha_\sigma - \delta^\beta_\sigma\delta^\alpha_\lambda\right)\right) & (42)\\ &= -\frac{F^{\beta\alpha}}{\mu_0} \, . & (43) \end{aligned}$$

The second term is $\frac{\partial \mathcal{L}}{\partial A_\alpha} = -J^\alpha$.

Therefore, the electromagnetic field's equations of motion are $\boxed{\frac{\partial F^{\beta\alpha}}{\partial x^\beta} \equiv \partial_\beta F^{\beta\alpha} = \mu_0 J^\alpha}$ which is the Poisson-Ampère equation. One recovers the Poisson equation for $\alpha = 0$.

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: Lorentz equation

To obtain the Lorentz equation, we need to use the Euler-Lagrange equation for the dynamic variables of particle positions. Only the Lagrangians associated with the particles and their interaction with the electromagnetic field are involved.

$$\boxed{\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta}$$ with $dt = \gamma d\tau$

For $\alpha = 0$, $$dE_{kin} = q\vec{E}\cdot\vec{u}dt$$

The variation in kinetic energy during the time interval *dt* is the work done by the electric field acting on the particle during this time interval.

For $\alpha = 1, 2, 3$ $$\boxed{\frac{d\vec{p}}{dt} = q\vec{E}(\vec{r}) + q\dot{\vec{r}} \times \vec{B}(\vec{r})}$$

According to Lorentz's equation, ***F* acts as an electromagnetic force**.

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: energy-stress tensor

➢ The stress-energy tensor is the conserved Noether current associated with space-time translations (external symmetries).

In special relativity the stress-energy tensor is the conserved Noether current associated with space-time translations. From the lagrangian density $\mathcal{L}$ we have

$$\boxed{T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial_\nu A_\alpha - \delta^{\mu}_{\nu} \mathcal{L}.}$$

from Noether*

$$\partial_\mu T^{\mu}_{\rho} = 0$$

$$\Rightarrow \quad \boxed{T^{\mu\nu} = \frac{1}{\mu_0} \left(\frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu}{}_{\lambda} F^{\lambda\nu} \right)}$$

* Note that this divergenceless property of this tensor is equivalent to four continuity equations

* See the paper: *A short review on Noether's theorems, gauge symmetries and boundary terms*

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: energy-stress tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(\frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu}{}_{\lambda}F^{\lambda\nu}\right) \longrightarrow T^{00} = \frac{1}{\mu_0}\left(\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + F^{0}{}_{\lambda}F^{\lambda 0}\right)$$

We have

$$\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{4}(F_{0i}F^{0i} + F_{i0}F^{i0}) + \frac{1}{4}F_{ij}F^{ij}$$

i=1,2,3

$$= -\frac{1}{2}\left(+\frac{E^i}{c}\frac{E^i}{c}\right) + \frac{1}{4}\times 2\times \vec{B}^2$$

Moreover

$$F^{0}{}_{\lambda}F^{\lambda 0} = F^{0}{}_{i}F^{i0} = -F^{0i}F^{i0} = (F^{i0})^2 = \frac{\vec{E}^2}{c^2} \qquad (*)$$

We deduce

$$\boxed{T^{00} = \frac{1}{\mu_0}\left(\frac{\vec{E}^2}{2c^2} + \frac{\vec{B}^2}{2}\right) = \frac{\epsilon_0\vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0}}$$

Then

$$T^{0i} = \frac{1}{\mu_0}\left(\frac{1}{4}\underbrace{g^{0i}}_{0}F_{\rho\sigma}F^{\rho\sigma} + F^{0}{}_{\lambda}F^{\lambda i}\right)$$

(*) $\eta_{\mu\nu} = \eta^{\mu\nu}$ and $\eta^{\nu}_{\mu} = \delta^{\nu}_{\mu}$.

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: energy-stress tensor

and

$$\begin{aligned} F^0{}_\lambda F^{\lambda i} &= F^0{}_j F^{ji} \\ &= -F^{0j} F^{ji} \\ &= F^{0j} F^{ij} \\ &= -\frac{E^j}{c}(-\epsilon_{ijk} B^k) \\ &= \frac{1}{c}(\vec{E} \wedge \vec{B})^i \end{aligned}$$

we deduce

$$\boxed{T^{0i} = \frac{1}{\mu_0 c}\left(\vec{E} \wedge \vec{B}\right)^i = \frac{\Pi^i}{c}}$$

We call the **Poynting** vector

$$\boxed{\vec{\Pi} = \frac{\vec{E} \wedge \vec{B}}{\mu_0}}$$

The electromagnetic stress-energy tensor can therefore be written as

$$\boxed{T^{\mu\nu} = \begin{pmatrix} u & \frac{\Pi^1}{c} & \frac{\Pi^2}{c} & \frac{\Pi^3}{c} \\ \frac{\Pi^1}{c} & T^{11} & T^{12} & T^{13} \\ \frac{\Pi^2}{c} & T^{21} & T^{22} & T^{23} \\ \frac{\Pi^3}{c} & T^{31} & T^{32} & T^{33} \end{pmatrix}}$$

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: energy-stress tensor

We set

$$\begin{cases} T^{ij} = \epsilon_0\left(-E^iE^j + \dfrac{\vec{E}^2}{2}\delta^{ij}\right) + \dfrac{1}{\mu_0}\left(-B^iB^j + \dfrac{\vec{B}^2}{2}\delta^{ij}\right) \\[2ex] u = \dfrac{1}{\mu_0}\left(\dfrac{\vec{E}^2}{2c^2} + \dfrac{\vec{B}^2}{2}\right) = \dfrac{\epsilon_0\vec{E}^2}{2} + \dfrac{\vec{B}^2}{2\mu_0} \end{cases}$$

energy density

We define the Maxwell tensor: $\boxed{\sigma_{ij} = -T_{ij}}$

In order to give a physical interpretation of this local relationship, we will integrate it over a finite volume Ω

$$\boxed{\partial_\mu T^\mu_\rho = 0}$$

No sources (only fields)

$$\frac{1}{c}\frac{\partial}{\partial t}T^{00} + \partial_i T^{i0} = 0$$

$$\frac{\partial}{\partial t}\int_\Omega T^{00}d^3\vec{x} + c\int_\Omega \partial_i T^{i0}d^3\vec{x} = 0$$

$$-\frac{\partial}{\partial t}\int_\Omega T^{00}d^3\vec{x} = c\int_\Omega \partial_i T^{i0}d^3\vec{x}$$

Stokes

$$-\frac{\partial}{\partial t}\int_\Omega T^{00}d^3\vec{x} = c\iint_{\partial\Omega} T^{0i}n^i dS$$

Since T^{00} is the energy density contained in the volume Ω, $cT^{i0} = \Pi^i$ is the outgoing energy flow.

This is the Poynting theorem

Covariant formulation of classical electromagnetism

Maxwell equations as a field theory: energy-stress tensor

Conservation laws in the presence of sources

$$\boxed{\partial_\mu T^\mu_\nu = \partial_\mu \left({}^{(field)}T^\mu_\nu + {}^{(part)}T^\mu_\nu \right) = 0}$$

$$\boxed{\frac{\partial}{\partial t}\left(\vec{P}_{part} + \int \frac{\Pi^i}{c^2} d^3\vec{x}\right) = -\int_\Omega \partial_j T^{ji} d^3\vec{x} = \int_\Omega \partial_j \sigma^{ji} d^3\vec{x} = \int_{\partial\Omega} \sigma^{ji} n_j dS}$$

This relation expresses that the variation of the total momentum contained in the Ω domain is equal to the sum of the forces exerted on the domain. It allows us to interpret

- $\frac{\vec{\Pi}}{c^2}$ as the local momentum density of the electromagnetic field
- $\sigma^{ji} n_j dS$ as the force exerted on a surface element $\vec{n}dS$ or σ is Maxwell's stress tensor

$$\boxed{\begin{gathered} {}^{(part)}T^{ik} = \mu c \bar{u}^i \bar{u}^k \frac{ds}{dt} \\ \mu = \sum_a m_a \delta(\vec{r} - \vec{r}_a) \end{gathered}}$$

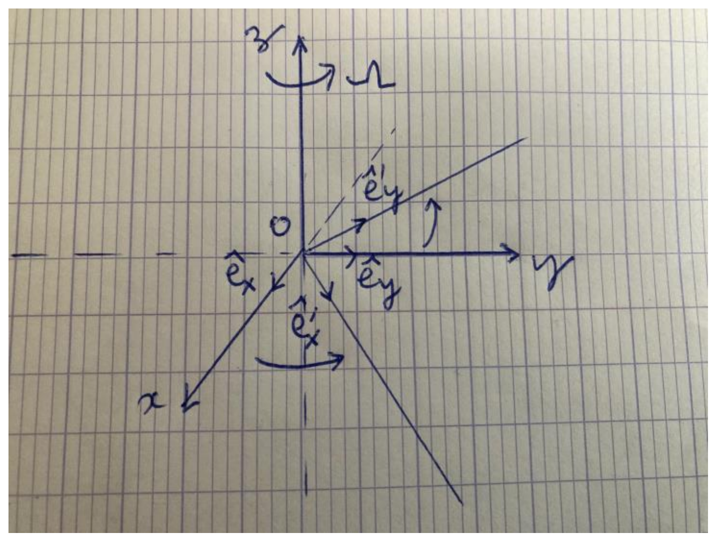
- ➢ Energy-stress tensor of a system of non-interacting particles
- ➢ μ is the mass density (can be continuous)

Gravitational field in relativistic mechanics

In an inertial reference frame based on Cartesian coordinates, the interval ds is determined by the formula

$$\boxed{ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2\ .}$$

When we move to any other inertial reference frame (i.e. using the Lorentz-Poincare transformation), we know that the expression of the space-time interval remains unchanged. But when we move to a non-inertial reference frame, the ds^2 is no longer the sum of the squares of the differentials of the four coordinates.



Example of a non-inertial reference frame:
uniformly rotating (around *z*-axis) coordinate system

Covariant formulation of classical electromagnetism

Gravitational field in relativistic mechanics

So when we move to a uniformly rotating coordinate system $x = x' \cos \Omega t - y' \sin \Omega t$, $y = x' \sin \Omega t + y' \cos \Omega t$, $z' = z$ the interval takes the form:

$$\boxed{ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt.}$$

Therefore, in a non-inertial frame of reference, the square of the interval is a certain general quadratic form of the coordinate differentials, i.e.

$$\boxed{ds^2 = \sum_{i,j=0..3} g_{ij} dx^i dx^j \ ,}$$

where the g_{ij} are functions of the space x^1, x^2, x^3 and time coordinates x^0. Thus, the four-coordinate system x^0, x^1, x^2, x^3 is curvilinear when using accelerated reference frames. The quantities g_{ij} that determine all the properties of geometry in each curvilinear coordinate system define the space-time metric.

Complements

Covariant formulation of classical electromagnetism

Maxwell's equations in terms of differential forms

Hodge dual or Hodge star operation.

Let us introduce an operation known as Hodge star which establishes a duality between k-forms and $(n-k)$-forms. Roughly speaking, it replaces exterior product of k variables by exterior product of the complementary set of $n-k$ variables (up to a constant factor, which depends on the metric tensor and the order of the variables in the two products). More precisely, let $\sigma = (i_1, i_2, ..., i_n)$ be a permutation of $(1, 2, ..., n)$ then for any $k \in \{0, 1, ..., n\}$ the Hodge dual of the corresponding elementary k-form is

$$\boxed{*(\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge ... \wedge \mathrm{d}x_{i_k}) = \mathrm{sgn}(\sigma)\varepsilon_{i_1}\varepsilon_{i_2}...\varepsilon_{i_k}\mathrm{d}x_{i_{k+1}} \wedge \mathrm{d}x_{i_{k+2}} \wedge ... \wedge \mathrm{d}x_{i_n}\ .} \tag{23}$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ and $(\varepsilon_{i_1}\varepsilon_{i_2}...\varepsilon_{i_k}) \in \{1, -1\}^n$ is the signature of the metric tensor. If we live in a Minkowski space-time with signature $(+, -, -, -)$ then $n = 4$ and $\varepsilon_t = -\varepsilon_x = -\varepsilon_y = -\varepsilon_z = 1$.

Examples

E_3 with the ordinary metric. If f and g are functions (0-form) of (x, y, z) one has:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \tag{24}$$

$$*\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}y \wedge \mathrm{d}z + \frac{\partial f}{\partial y}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial f}{\partial z}\mathrm{d}x \wedge \mathrm{d}y\ . \tag{25}$$

Covariant formulation of classical electromagnetism

Maxwell's equations in terms of differential forms

$$*1 = dt \wedge dx \wedge dy \wedge dz, \qquad *(dt \wedge dx \wedge dy \wedge dz) = 1 \cdot (-1)^3 = -1.$$

$$\begin{aligned} *(dt) &= dx \wedge dy \wedge dz, \\ *(dx) &= dt \wedge dy \wedge dz, \\ *(dy) &= dt \wedge dx \wedge dz, \\ *(dz) &= dt \wedge dx \wedge dy. \end{aligned}$$

$$\begin{aligned} *(dt \wedge dx) &= dz \wedge dy, & *(dz \wedge dy) &= -dt \wedge dx, \\ *(dt \wedge dy) &= dx \wedge dz, & *(dx \wedge dz) &= -dt \wedge dy, \\ *(dt \wedge dz) &= dy \wedge dx, & *(dy \wedge dx) &= -dt \wedge dz. \end{aligned}$$