M1

General Relativity

Exam, December 2024, duration 2 hours

Exercises are independent, VS: very simple, S: simple, M: medium

1 Eddington's grand expedition [VS]

Explain the Eddington's grand expedition. What was its purpose? How was it carried out? What phenomenon was Eddington trying to prove?

2 Laws of physics in tensor form [VS]

(i) Why is it essential to write the laws of physics in tensor form? (ii) Demonstrate the covariance of Maxwell's equation written in tensor form $\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$. Hint: use the Λ matrix for the Lorentz-Poincaré transformation. (iii) The Lorentz equation $\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta$ is covariant in special relativity, i.e. with respect to the Lorentz-Poincaré transformation. What is its expression in general relativity or in an non-inertial reference frame characterised by a metric tensor g_{ij}?

3 Einstein's equation [S]

Explain (using your own words) how to obtain Einstein's equation $R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik}$ of general gravitation in **two different ways**. How can we justify (mathematically and historically) the addition of the cosmological constant? Why Albert Einstein introduced this constant?

4 Image of a black hole [S]

In 2017, the Event Horizon Telescope took the first image of a black hole. This object called, M87*, is the beating heart of the giant elliptical galaxy Messier 87 and lives 55 million light years away from Earth. The image of the black hole revealed a bright circular ring. Explain the origin of this phenomenon.

5 Near-horizon orbits [M]

For a given mass m', the radius $r_g = \frac{2Gm'}{c^2}$ is called the Schwarzschild radius. When the orbit radius r is large relative to r_g ($r \gg r_g$), the only relativistic effects are small corrections to Kepler orbits (e.g. Mercury's perihelion). Closer to r_g, interesting things happen. At shorter distances from r_g, relativistic effects are much stronger. To study them, let's reconsider the effective potential:

$$V_{\text{eff}}(r) = -\frac{Gmm'}{r} + \frac{L^2}{2mr^2} - \frac{Gm'L^2}{c^2 m}\frac{1}{r^3} ,$$

where m is the mass of the orbiting object and L it's orbital angular momentum. Its plot is given in Lesson 6. We can see that there is a second extremum absent in the Newtonian case. It represents a situation where the force becomes attractive again, overcoming the centrifugal force due to angular momentum.

1. Find the expression for the radii of these two extrema, which we'll call $r_\pm$. We will write $r_\pm = \frac{a\pm\sqrt{b}}{c}$.
2. For a sufficiently large angular momentum L, the square root $\sqrt{b}$ is positive and the potential admits two extrema. The largest extremum (at r_+) is a minimum that determines the stable Kepler circular orbit. The second extremum (at r_-) corresponds to a maximum of the effective potential. It is therefore an unstable lower circular orbit. We now want to find the lowest stable orbit. Below a certain value of L the square root becomes negative. Find the value of this angular momentum, L^*.
3. For this critical value of $L = L^*$ we have $r_- = r_+ = r_{min}$. Give the expression of r_{min} as a function of r_g. There are no stable orbits below r_{min}.
4. What happens when objects reach this distance?
5. Calculate r_{min} for a satellite of mass $m = 10^4$ kg orbiting around a black hole of intermediate mass $m' = m_{bh} = 10^5$ $M_\odot$. Express r_{min} as a function of $R_{bh} = 10^3$ km, the radius of the black hole. Perform the same calculation for a stellar black hole $m' = m_{bh} = 10$ $M_\odot$ and $R_{bh} = 30$ km. Conclusion?

Note: $M_\odot = 2 \times 10^{30}$ kg.

6 Riemann spaces 2 [M]

Consider the two-dimensional space-time with line element

$$ds^2 = dv^2 - v^2 du^2 \ .$$

1. Write the metric tensor.
2. Compute the Christoffel symbols. How many components?
3. Compute the non-vanishing components of the curvature (Riemann) tensor.
4. Conclusion.

7 Covariant derivative [S-M, Bonus]

The covariant derivative of a contravariant vector U^μ is: $DU^\mu = \left(\partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda\right) dx^\nu$. Use this expression to obtain the covariant derivative of the covariant vector U_μ, i.e. DU_μ.

8 Christoffel symbols [M, Bonus]

Show that Γ^i_{kl} is not a tensor.