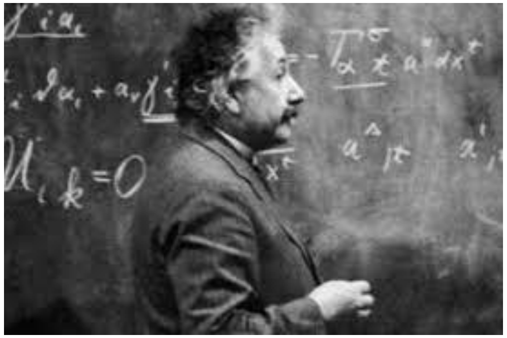
# General Relativity (GR)

M1 - Physique 2025-2026



AE+GR (1907-1917)

*Paul-Antoine Hervieux*
*Unistra/IPCMS*
*hervieux@unistra.fr*

$$g_{\mu\nu}$$

$$S[g] = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\lambda)d^4x$$

## *III*) Theory: [$g_{ij}$] link between geometry and gravity

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 8\pi G T_{ab}$$

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0$$

# GR Field equations

**"Geometry tells matter how to move, and matter tells geometry how to curve"**

To find the equations determining a gravitational field, one must first determine the action $S_g$ for this field. The equations are then obtained by varying the sum of the field and particle actions with respect to $g_{ij}$.

*Field equations*

- The $S_g$ action, like the electromagnetic action $S_{field}$, must be expressed as a scalar integral $\int H\sqrt{-g}d\Omega$ extended over all space and between two values of the time component $x^0$. We remind you that $d\Omega = dx^0dx^1dx^2dx^3 = cdtdV$ and $g = -1$ for SR. $g \equiv [g_{ij}]$
- The gravitational field equations must not contain derivatives of field potentials of order greater than 2, as is the case for the electromagnetic field equations ($A_\alpha$ plays the role of $g_{ij}$): $\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$ with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.
- Since the field equations are obtained by varying the action, the $H$ expression must not contain derivatives of $g_{ik}$ greater than one. So $H$ must contain only the tensor $\mathbb{G}$ and the symbols $\Gamma$.
- Unfortunately, the quantities $g_{ik}$ and $\Gamma^i_{kl}$ alone cannot be used to form a scalar.

# GR Field equations

- There is, however, a scalar $R$ - the curvature of 4-space - which admittedly contains the metric tensor $\mathbb{G}$ and its first and second derivatives, but only linearly.

It is possible to simplify the expression of $S_g$ by eliminating the second derivatives $\partial^2 g_{ik}/\partial x^p \partial x^q$ involved in the $R$ invariant. (*)

After this simplification, the integrand of $S_g$ will contain only the tensor $\mathbb{G}$ and the Christoffel symbols $\Gamma$. We have

$$\delta \int R\sqrt{-g}d\Omega = \delta \int H\sqrt{-g}d\Omega \ , \tag{1}$$

with

$$H(g, \partial g) = g^{ik}(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}) \ . \tag{2}$$

(*) The demonstration is quite difficult. It can be found in "Modern Geometry - Methods and Applications", Part I. The Geometry of Surfaces, Transformation Groups, and Fields (B. A. Dubrovin , A. T. Fomenko , S. P. Novikov). The main tool is the Stokes' theorem.

# GR Field equations

We can therefore write:

relativity — curvature of space — gravitation

$$\delta S_g = -\frac{c^3}{16\pi G}\delta\int H\sqrt{-g}d\Omega = -\frac{c^3}{16\pi G}\delta\int R\sqrt{-g}d\Omega\ , \qquad (3)$$

where $G$ is called the universal gravitational constant measured in N.m$^2$/kg$^2$. Verify that $S_g$ has the dimensions of an action.

lagrangian density

Starting from $\delta S_g = -\frac{c^3}{16\pi G}\delta\int(H\sqrt{-g})d\Omega$ we can easily obtain the Euler-Lagrange equations as

$$\delta S_g = -\frac{c^3}{16\pi G}\int\left\{\frac{\partial(H\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(H\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik}d\Omega\ . \qquad (4)$$

After calculation, one finds:

$$\frac{1}{\sqrt{-g}}\left\{\frac{\partial(H\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(H\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\} = \boxed{R_{ik} - \frac{1}{2}g_{ik}R\ .} \qquad (5)$$

# GR Field equations

*Action of the matter*

matter

$$\delta S_m = \frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega \;, \qquad (6)$$

where $T_{ik}$ is the stress-energy tensor of matter. Gravitational interaction only comes into play for bodies with sufficiently large masses. The expression $T_{ik} = (p+\epsilon)\overline{u}_i\overline{u}_k - pg_{ik}$ should be used, where $\epsilon$ is the energy density of the body ($\epsilon/c^2$ is the mass density, i.e. the mass of the body's "proper" unit volume, i.e. the volume in the frame of reference where the body element is at rest) and $p$ is the pressure

Note: in the presence of electromagnetic fields $T_{ik}$ is the sum of the total stress-energy tensor including the one coming from the electromagnetic field.

# GR Field equations

$$S_m = \frac{1}{c}\int \Lambda\sqrt{-g}d\Omega \longrightarrow \delta S_m = \frac{1}{c}\int\left\{\frac{\partial(\Lambda\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(\Lambda\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik}d\Omega\ . \quad (7)$$

If we define

$$\frac{1}{2}\sqrt{-g}T_{ik} = \frac{\partial(\Lambda\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(\Lambda\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\ , \quad (8)$$

one get's

$$\delta S_m = \frac{1}{2c}\int\sqrt{-g}T_{ik}\delta g^{ik}d\Omega\ . \quad (9)$$

We have used that $T^k_{i;k} = 0$: the divergence of the stress-energy tensor is zero.

→ Conservation law of energy/momentum (Noether)

*Einstein tensor*

We start from the Bianchi identity ($\nabla_t R^l_{irs} \equiv R^l_{irs;t}$)

$$\nabla_t R^l_{irs} + \nabla_s R^l_{itr} + \nabla_r R^l_{ist} = 0\ . \quad (10)$$

First group of Maxwell equations

$$F_{\alpha,\beta;\gamma} + F_{\beta,\gamma;\alpha} + F_{\gamma,\alpha;\beta} = 0$$

$dF = 0$

According to their tensor form, this second Bianchi identity is valid in any coordinate system and at any point of the Riemann space.

# GR Field equations

Performing a first contraction on the Bianchi identity for $t = l$, we obtain

$$\nabla_l R^l_{irs} + \nabla_s R^l_{ilr} + \nabla_r R^l_{isl} = 0 \; . \tag{11}$$

Taking into account the definition $R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}$ of the Ricci tensor and the equality $R^l_{rsl} = -R^l_{rls}$ one obtains

$$\longrightarrow \qquad \nabla_l R^l_{irs} + \nabla_s R_{ir} - \nabla_r R_{is} = 0 \; . \tag{12}$$

Since the change in variance by means of $g_{ik}$ is permutable with the covariant we have:

$$\longrightarrow \qquad \nabla_s R^k_r = \nabla_s g^{ik} R_{ir} = g^{ik} \nabla_s R_{ir} \; . \quad (\text{since} \quad \nabla_s g^{ik} = 0) \tag{13}$$

Multiply relation (12) by $g^{ik}$ and use the permutation property (13), we obtain

$$\longrightarrow \qquad \nabla_l R^{kl}_{rs} + \nabla_s R^k_r - \nabla_r R^k_s = 0 \; . \tag{14}$$

Performing a second contraction with respect to indices $k$ and $s$, we get:

$$\longrightarrow \qquad \nabla_l R^{kl}_{rk} + \nabla_k R^k_r - \nabla_r R^k_k = 0 \; . \tag{15}$$

# GR Field equations

After contraction, let's change the summation index $l$ to index $k$ in the first term of the previous equation. Moreover, with the equality $R^{kl}_{rk} = R^{lk}_{kr}$ we obtain

$$2\nabla_k R^k_r - \nabla_r R = 0 \; . \tag{16}$$

The latter expression can also be written as:

$$\nabla_k \left( R^k_r - \frac{1}{2}\delta^k_r R \right) = \nabla_k S^k_r = 0 \; . \tag{17}$$

The expression in brackets in the previous expression is a tensor, denoted $S^k_r$, whose covariant components are given by:

$$S_{ij} = g_{ik} S^k_j = g_{ik} \left( R^k_j - \frac{1}{2}\delta^k_j R \right) = R_{ij} - \frac{1}{2} g_{ij} R \; . \tag{18}$$

The $S_{ij}$ tensor is called the Einstein tensor. Due to the symmetry of the Ricci tensor, the Einstein tensor is also symmetrical. According to (17), it verifies the identities:

$$\nabla_k S^k_r = 0 \; . \tag{19}$$

A tensor that satisfies relations of the form (19) identically is called a conservative tensor.

# GR Field equations

Elie Cartan showed that if both of the following conditions apply:

i) the quantities $S_{ij}$ depend only on the gravitational potentials $g_{ij}$ and their first- and second-order derivatives, and are linear with respect to the second-order derivatives; ii) the tensor $S_{ij}$ satisfies the conservation equations $\nabla_k S^k_r = 0$, then the only tensors satisfying both above conditions are given by the formula

$$S_{ij} = h\left[R_{ij} - \frac{1}{2}g_{ij}(R+k)\right] \;, \tag{20}$$

where $h$ and $k$ are two constants.

Elie Cartan

(1869-1951)

# Vacuum equations

*Vacuum equations*

In GR, a region where $T_{ik} = 0$, i.e. no matter, is called a "vacuum". Einstein's equations are

$$\boxed{R_{ij} - \frac{1}{2} R g_{ij} = 0 \ .} \tag{21}$$

By contracting with $g^{ij}$ we obtain $R = 0$ ($g_{ij} g^{ij} = \delta^i_i = 4$). Reintegrating into (21) gives:

$$\boxed{R_{ij} = 0 \ .} \tag{22}$$

In other words, in the absence of matter, the Ricci tensor is zero. This is not a sufficient condition for flatness, however, so in general space-time can be curved even in the absence of matter. Matter is therefore not enough to determine the gravitational field, just as charges are not enough to determine the electromagnetic field. A space-time in which the Ricci tensor is zero everywhere is called Einstein space-time. Therefore:

$$\boxed{\mathrm{Riemann\ \ tensor} = 0 \leftrightarrow \mathrm{flat\ \ space} \ \ ; \ \ \mathrm{Ricci\ \ tensor} = 0 \leftrightarrow \mathrm{empty\ \ space} \ .} \tag{23}$$

These equations define general relativity. They are sufficient to describe gravitational waves, black holes, the expansion of the universe and the Big Bang, to found GPS technology...

# Einstein equations with matter

*Einstein equations with matter*

so from the principle of least action we deduce $\delta S_m + \delta S_g = 0$ leading to

$$-\frac{c^3}{16\pi G}\int\left(R_{ik} - \frac{1}{2}g_{ik}R - \frac{8\pi G}{c^4}T_{ik}\right)\delta g^{ik}\sqrt{-g}d\Omega = 0\ , \tag{24}$$

hence, given that $\delta g^{ik}$ are arbitrary one obtains

$$\boxed{R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik}\ ,} \tag{25}$$

or

$$\boxed{R_i^k - \frac{1}{2}g_i^k R = \frac{8\pi G}{c^4}T_i^k\ .} \qquad g_i^k = \delta_i^k \tag{26}$$

These are the gravitational field equations or Einstein equations. We have used that $g_{il}g^{lk} = \delta_i^k$. $g^{ik}$ is the inverse of $g_{ik}$.

# Cosmological constant

Contracting on the indices $i$ and $k$ we find ($\delta^i_i = 4,\ R \equiv R^i_i,\ T \equiv T^i_i$)

$$R = -\frac{8\pi G}{c^4} T\ . \tag{27}$$

Consequently, the field equations can be recopied as

$$\boxed{R_{ik} = \frac{8\pi G}{c^4}\left(T_{ik} - \frac{1}{2} g_{ik} T\right)\ .} \tag{28}$$

Note: The equations of the gravitational field are not linear. As a result, the principle of superposition is not true for gravitational fields, as it is for the electromagnetic field in SR. However, it should be borne in mind that we're generally dealing with weak gravitational fields whose equations are linear to a first approximation. So the superposition principle remains legitimate with the same approximation.

Note: From (20) Einstein's equation can be rewritten as

$$\boxed{\nabla_k S^k_r = 0 \leftrightarrow \nabla_k T^k_r = 0}$$

$$\boxed{R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}\ .} \tag{29}$$

$\Lambda$ is called the cosmological constant.

# RG with differential forms

$$S_g = \frac{1}{2\chi}\int R^{ab}\wedge{}^{\star}(V_a\wedge V_b).$$

Curvature 2-form: $R^{ab} = \frac{1}{2}R^{ab}_{\mu\nu}dx^\mu\wedge dx^\nu$

By using the *vierbeins* $V^a_\mu$ we can introduce in the tangent Minkowski space-time a set of basis 1-forms $V^a = V^a_\mu dx^\mu$.

*Proof*:

$$\begin{aligned} R^{ab}\wedge{}^{\star}(V_a\wedge V_b) &= \frac{1}{2}R_{\mu\nu}{}^{ab}\frac{1}{2}V^\alpha_a V^\beta_b\,\eta_{\alpha\beta\rho\sigma}\,dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma \\ &= \frac{1}{4}R_{\mu\nu}{}^{ab}V^\alpha_a V^\beta_b\,\eta_{\alpha\beta\rho\sigma}\eta^{\mu\nu\rho\sigma}\,d^4x\,\sqrt{-g} \\ &= -\frac{1}{2}R_{\mu\nu}{}^{ab}V^\alpha_a V^\beta_b\left(\delta^\mu_\alpha\delta^\nu_\beta-\delta^\nu_\alpha\delta^\mu_\beta\right)d^4x\,\sqrt{-g} \\ &= -R\,d^4x\,\sqrt{-g} \end{aligned}$$

# Stress-energy tensor

*Energy-momentum tensor of a set of independent particles (no interaction between them) of masses $m_a$.*

We define the mass density as

$$\mu = \sum_a m_a \delta(\vec{r} - \vec{r}_a) \ , \tag{35}$$

where $\vec{r}_a$ are the radius vectors of the particles. We have:

$$T^{ik} = \mu c \bar{u}^i \bar{u}^k \frac{ds}{dt} = \mu c^2 \sqrt{1-\beta^2} \bar{u}^i \bar{u}^k \ . \tag{36}$$

*Conservation law*

We verify that

$$\partial_k \left( T^{(field)}{}_i^k + T^{(particles)}{}_i^k \right) = 0 \ . \tag{37}$$

The meaning of this relation is that the sum of the energies and momenta of the field-particles are conserved.

# Stress-energy tensor

*Energy-momentum tensor for macroscopic bodies.*

In addition to the energy-momentum (or stress-energy) tensor for a system of point particles we shall also need the expression for this tensor for macroscopic bodies which are treated as being continuous. We have

$$T^{ik} = (p+\epsilon)\bar{u}^i\bar{u}^k - pg^{ik} \ , \tag{38}$$

or

$$T^{ik} = (p+\epsilon)\bar{u}^i\bar{u}^k - p\delta^{ik} \ . \tag{39}$$

In the above expression $\epsilon \equiv \rho c^2$ is the energy density, $p$ the pressure and $\bar{u}^i = \frac{\gamma}{c}\frac{dx^i}{dt}$ the dimensionless velocity.

*Conservation law*

$$T^{ik}_{fluid} = (\rho + \frac{p}{c^2})u^i u^k - p\delta^{ik} \ . \tag{40}$$

When $v^2 \ll c^2$ one has $\nabla_\alpha T^{\alpha\beta}_{fluid} = 0 \rightarrow$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{41}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = \frac{1}{\rho}\vec{\nabla}p + \vec{F}_{ext} \ . \tag{42}$$

The first equation is the continuity equation and is related to the mass conservation. The other three are linked to the Euler equation of fluid mechanics (conservation of momentum).

# Einstein's approach

- Generalization of the Poisson and Laplace equations.

$$\boxed{\Delta\phi = 4\pi G\rho} \quad \text{(o)}$$

$\phi$ is the gravitational field produced by the mass density $\rho$

- The equations must be expressed as relations between tensors in space-time.

$$\text{(o)} \quad \Leftrightarrow \boxed{S_{\lambda\mu} = \chi T_{\lambda\mu}} \quad \text{(oo)} \qquad \chi \text{ : constant}$$

- $T_{\lambda\mu}$ is purely mechanical and describes the state of energy and matter distribution at each point. It generalizes the second member of Poisson's equation.
- $S_{\lambda\mu}$ generalizes the left member of the Poisson equation. It is a purely geometrical quantity. By analogy with $\Delta\phi$ it contains gravitational potentials and their first- and second-order derivatives.
- We know that $\nabla_\mu T^{\mu\nu} = 0$. This is the law of conservation of energy and momentum.

$$\text{(oo)} \quad \Rightarrow \boxed{\nabla_\mu S^{\mu\nu} = 0}$$