Problem Set 1
Scattering in 1d

Disclaimer: The following problems (as well as the figures) are reproduced and/or adapted from the excellent textbook by D.J. Griffiths, *Introduction to Quantum Mechanics* (Cambridge University Press). Several copies of the book can be found at the University library.

Problem 1.1

Consider a nonrelativistic quantum particle of mass m subject to the double δ-function potential

$$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right], \tag{1}$$

where α and a are positive constants.

(a) Sketch the potential (1).

(b) Let us first consider bound states ($E < 0$). How many bound states does the potential (1) possess, depending on the value of α?
Hint 1: Remember that for an even potential $V(x) = V(-x)$, the wave functions can always be taken to be either even $[\psi(x) = \psi(-x)]$ or odd $[\psi(x) = -\psi(-x)]$.
Hint 2: Sometimes, graphical solutions to transcendental equations and computers may be useful!

(c) Find the allowed energies for (i) $\alpha = \hbar^2/ma$ and (ii) $\alpha = \hbar^2/4ma$. Sketch the corresponding wave functions.

(d) What are the bound state energies in the limiting cases (i) $a \to 0$ and (ii) $a \to \infty$ (holding α fixed)? Explain why your answers are reasonable, by comparison with the single δ-function well discussed in the lecture.

(e) We now look at scattering states ($E > 0$). Determine the transmission coefficient for the potential (1).

Problem 1.2

Analyze the *odd* bound state wave functions for the finite square well discussed in the lecture. Derive the transcendental equation for the allowed energies, and solve it graphically. Discuss the two limiting cases of a wide, deep well as well as a shallow, narrow well. Is there always on odd bound state?

Problem 1.3

We consider the rectangular barrier

$$V(x) = \begin{cases} V_0, & -a < x < +a, \\ 0, & |x| > a, \end{cases}$$

with both V_0 and a positive constants.

(a) Show that the corresponding transmission coefficient T is given by

$$T = \left\{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)\right\}^{-1}$$

for $0 \leqslant E < V_0$.

(b) What is the transmission coefficient for $E > V_0$.
Hint: If you think carefully about it, there is hardly anything to do to get T in that case.

(c) How much is T when $E = V_0$?

(d) Plot T as a function of E (you are allowed to use a computer!)

Problem 1.4

Consider the "step" potential

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x > 0, \end{cases}$$

with $V_0 > 0$.

(a) For a quantum particle coming from left to right, calculate the reflection coefficient R for the case $E < V_0$ and comment on the answer.

(b) For the case $E > V_0$, show that

$$\frac{\left(\sqrt{E} - \sqrt{E - V_0}\right)^4}{V_0^2}.$$

(c) Sketch R as a function of energy.

(d) For a potential barrier such as this, which does not go back to zero to the right of the barrier, the transmission coefficient T is *not* simply $|F|^2/|A|^2$ (with A the incident amplitude and F the transmitted amplitude), because the transmitted wave travels at a different *speed*. Show that

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|F|^2}{|A|^2}$$

for $E > V_0$.
Hint: Determine the incident and transmitted probability currents, J_i and J_t, respectively, and use the fact that $T = J_\mathrm{t}/J_\mathrm{i}$.

(e) For $E > V_0$, calculate T for the step potential and check that $T + R = 1$.

Problem 1.5

A particle of mass m and kinetic energy $E > 0$ approaches an abrupt potential drop V_0:

$$V(x) = \begin{cases} 0, & x < 0, \\ -V_0, & x > 0, \end{cases}$$

with $V_0 > 0$. What is the probability that it will "reflect" back if $E = V_0/3$?

Problem 1.6

The theory of scattering generalizes in a pretty obvious way to arbitrary localized potentials, such as the one of Fig. 1.

To the left (Region I), $V(x) = 0$, so

$$\psi(x) = A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx},$$

where $k = \sqrt{2mE}/\hbar$. To the right (Region III), $V(x) = 0$ also, so

$$\psi(x) = F\,\mathrm{e}^{\mathrm{i}kx} + G\,\mathrm{e}^{-\mathrm{i}kx}.$$

In between (Region II), of course, we cannot know precisely what ψ is unless the potential is specified. But because the Schrödinger equation is a linear, second-order differential equation, the general solution has got to be of the form

$$\psi(x) = Cf(x) + Dg(x),$$

where $f(x)$ and $g(x)$ are two linearly independent particular solutions.

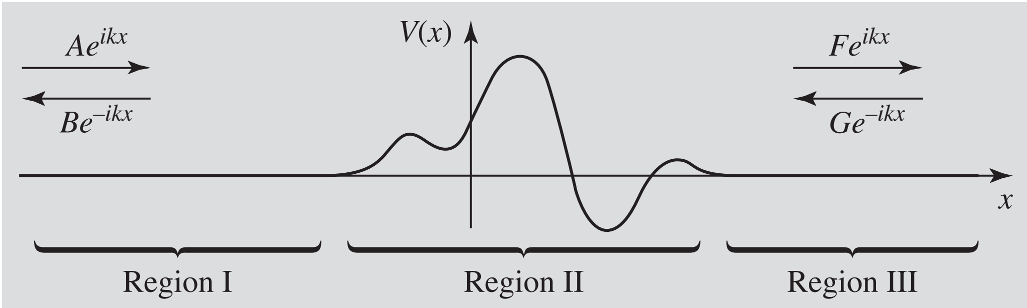

Figure 1: Scattering from an arbitrary localized potential [$V(x) = 0$, except in Region II].

There will be four boundary conditions (two joining Regions I and II, and two joining Regions II and III). Two of these can be used to eliminate C and D, and the other two can be "solved" for B and F in terms of A and G:

$$B = S_{11}A + S_{12}G,$$
$$F = S_{21}A + S_{22}G.$$

The four coefficients S_{ij}, which depend on k (and hence on E), constitutes a 2×2 matrix $\boldsymbol{S}$, called the **scattering matrix** (or **$\boldsymbol{S}$-matrix** for short). The S-matrix tells you the outgoing amplitudes (B and F) in terms of the incoming amplitudes (A and G):

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix} \tag{2}$$

In the typical case of scattering from the left, $G = 0$, so the reflection and transmission coefficients are

$$R_{\mathrm{L}} = \frac{|B|^2}{|A|^2} = |S_{11}|^2, \qquad T_{\mathrm{L}} = \frac{|F|^2}{|A|^2} = |S_{21}|^2. \tag{3}$$

For scattering from the right, $A = 0$, and

$$R_{\mathrm{R}} = \frac{|F|^2}{|G|^2} = |S_{22}|^2, \qquad T_{\mathrm{R}} = \frac{|B|^2}{|G|^2} = |S_{12}|^2. \tag{4}$$

(a) Construct the S-matrix for scattering from a δ-function well.

(b) Construct the S-matrix for the finite square well.

Hint: To answer these two questions, you may want to have a look to your lecture notes. There is hardly anything to do! On top of that, for question (b), be lazy and think of the symmetry of the problem.

Problem 1.7

The S-matrix of the previous problem tells you the *outgoing* amplitues (B and F) in terms of the *incoming* amplitudes (A and G), see Eq. (2). For some purposes it is more convenient to work with the **transfer matrix $\boldsymbol{M}$**, which gives you the amplitude to the *right* of the potential (F and G) in terms of those to the *left* (A and B):

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

(a) Find the four elements of the M-matrix in terms of the elements of the S-matrix, and *vice versa*. Express R_L, T_L, R_R, and T_R [*cf.* Eqs. (3) and (4)] in terms of elements of the M-matrix.

(b) Suppose you have a potential consisting of two isolated pieces (Fig. 2). Show that the M-matrix for the combination is the *product* of the two M-matrices for each section separately, *i.e.*,

$$\boldsymbol{M} = \boldsymbol{M}_2\boldsymbol{M}_1.$$

Note that the above result generalizes to any number of pieces, which accounts for the usefulness of the M-matrix formalism.

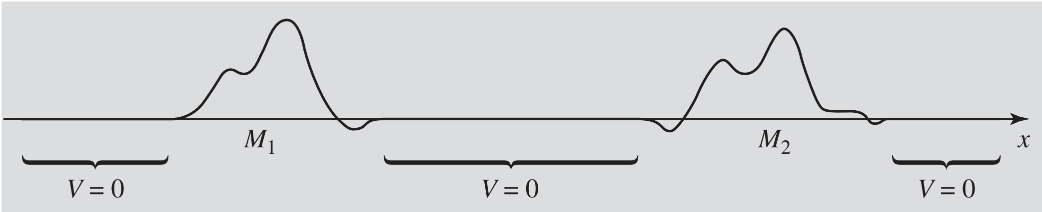

Figure 2: A potential consisting of two isolated pieces.

(c) Construct the M-matrix for scattering from a single δ-function potential at point a,

$$V(x) = -\alpha\delta(x-a),$$

with $\alpha > 0$.

(d) By the method of Question (b), find the M-matrix for scattering from the double δ-function of Eq. (1). What is the transmission coefficient for such a potential?