# Problem Set 2
# The WKB approximation

Disclaimer: The following problems (as well as the figures) are reproduced and/or adapted from the excellent textbook by D.J. Griffiths, *Introduction to Quantum Mechanics* (Cambridge University Press). Several copies of the book can be found at the University library.

## Problem 2.1

Let us consider the following square well potential with a "shelf" of height $V_0 > 0$, extending half-way accross:

$$V(x) = \begin{cases} V_0, & 0 < x < a/2, \\ 0, & a/2 < x < a, \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

(a) Sketch the potential (1).

(b) Consider first the case in which $V_0 = 0$. Show that the allowed eigenenergies are given by

$$E_n^0 = \frac{\pi^2 \hbar^2}{2ma^2} n^2, \tag{2}$$

with $n \in \mathbb{N}^*$. Find the corresponding wave functions.

(c) Consider now the case $V_0 \neq 0$. Use the WKB approximation to find the allowed energies $E_n$ of the potential (1). Express your answer in terms of $E_n^0$ [*cf.* Eq. (2)].

(d) Using perturbation theory to first order, find the energies $E_n^1$ of the potential (1).

(e) Assume that $E_1^0 > V_0$, but do *not* assume than $E_n \gg V_0$. Compare your WKB result with what you got in the previous question using first-order perturbation theory. Note that they are in agreement if either $V_0$ is very small (the perturbation theory regime) or $n$ is very large (the WKB, semiclassical regime).

## Problem 2.2

An illuminating alternative derivation of the WKB formula

$$\psi(x) \simeq \frac{C}{\sqrt{p(x)}} \exp\left(\pm\frac{\mathrm{i}}{\hbar}\int \mathrm{d}x\, p(x)\right), \tag{3}$$

with

$$p(x) = \sqrt{2m\,[E - V(x)]}, \tag{4}$$

is based on an expansion in powers of $\hbar$.

(a) Show that the time-independent Schrödinger equation can be put in the form

$$\psi'' = -\frac{p^2}{\hbar^2}\,\psi. \tag{5}$$

(b) Motivated by the free-particle wave function $\psi = A\exp\left(\pm \mathrm{i}px/\hbar\right)$, let us write

$$\psi = \mathrm{e}^{\mathrm{i}f(x)/\hbar}, \tag{6}$$

where $f(x)$ is some *complex* function. Put the ansatz (6) into Eq. (5) to show that

$$\mathrm{i}\hbar f'' - \left(f'\right)^2 + p^2 = 0.$$

(c) Write $f(x)$ as a power series in $\hbar$:

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \dots$$

By collecting like powers in $\hbar$, show that

$$\left(f_0'\right)^2 = p^2, \qquad \mathrm{i}f_0'' = 2f_0' f_1', \qquad \mathrm{i}f_1'' = 2f_0' f_2' + \left(f_1'\right)^2, \qquad \textit{etc.}$$

(d) Solve for $f_0(x)$ and $f_1(x)$, and show, to first order in $\hbar$, that you recover Eq. (3).

## Problem 2.3

In the lecture, we have discussed that the transmission coefficient $T$ for a potential barrier of length $a$ is given by

$$T = \mathrm{e}^{-2\gamma}, \tag{7a}$$

with

$$\gamma = \frac{1}{\hbar}\int_0^a \mathrm{d}x\, |p(x)|\,, \tag{7b}$$

where $p(x)$ is given in Eq. (4).

(a) Use Eq. (7) to calculate the approximate transmission probability for a particle of energy $E$ that encounters a finite square barrier of height $V_0 > E$ and width $2a$.

(b) Compare your answer with the exact result, to which it should reduce in the WKB regime $T \ll 1$.
Hint: You may want to have a look to Problem Set 1 to answer this.

## Problem 2.4

(a) Use the WKB approximation to find the allowed energies of the harmonic oscillator (mass $m$, angular frequency $\omega$).
Hint:

$$\int \mathrm{d}u\,\sqrt{1-u^2} = \frac{1}{2}\left(u\sqrt{1-u^2} + \arcsin u\right) + \mathrm{cst}.$$

(b) Let us consider from now on that the quantum particle is in the $n^{\mathrm{th}}$ stationary state of the harmonic oscillator. Find the corresponding turning point $x_2$.

(c) How far (call the distance $d$) could you go *above* the turning point before the error in the linearized potential reaches $1\,\%$? That is, if

$$\frac{V(x_2+d) - V_{\mathrm{lin}}(x_2+d)}{V(x_2)} = 0.01,$$

what is $d$?

(d) The asymptotic form of the Airy function $\mathrm{Ai}(z)$ is accurate to $1\,\%$ as long as $z \geqslant 5$. For the $d$ in Question (c), determine the smallest $n$ such that $\alpha d \geqslant 5$, with $\alpha = [2mV'(x_0)/\hbar^2]^{1/3}$. (For any $n$ larger than this there exists an overlap region in which the linearized potential is good to $1\,\%$ *and* the large-$z$ form of the Airy function is good to $1\,\%$.)

## Problem 2.5

In the lecture, the connection formulas at a downward-sloping turning point was given to be

$$\psi(x) \simeq \begin{cases} \dfrac{D'}{\sqrt{|p(x)|}} \exp\left(-\dfrac{1}{\hbar}\displaystyle\int_x^{x_1} \mathrm{d}x'\, |p(x')|\right), & x < x_1, \\ \dfrac{2D'}{\sqrt{p(x)}} \sin\left(\dfrac{1}{\hbar}\displaystyle\int_{x_1}^{x} \mathrm{d}x'\, p(x') + \dfrac{\pi}{4}\right), & x > x_1. \end{cases} \tag{8}$$

Derive Eq. (8).

## Problem 2.6

Use appropriate connection formulas to analyze the problem of scattering from a barrier with sloping walls (see Fig. 1).

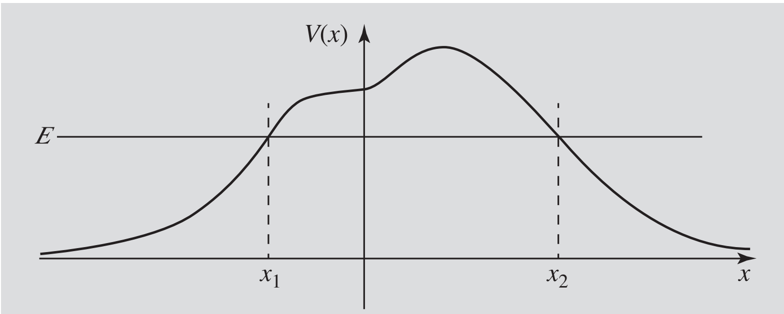


Figure 1: Barrier with sloping walls.

Hint: Begin by writing the WKB wave function in the form

$$\psi(x) \simeq \begin{cases} \dfrac{1}{\sqrt{p(x)}} \left[A \exp\left(-\dfrac{\mathrm{i}}{\hbar}\displaystyle\int_x^{x_1} \mathrm{d}x'\, p(x')\right) + B \exp\left(\dfrac{\mathrm{i}}{\hbar}\displaystyle\int_x^{x_1} \mathrm{d}x'\, p(x')\right)\right], & x < x_1, \\ \dfrac{1}{\sqrt{|p(x)|}} \left[C \exp\left(\dfrac{1}{\hbar}\displaystyle\int_{x_1}^{x} \mathrm{d}x'\, |p(x')|\right) + D \exp\left(-\dfrac{1}{\hbar}\displaystyle\int_{x_1}^{x} \mathrm{d}x'\, |p(x')|\right)\right], & x_1 < x < x_2, \\ \dfrac{1}{\sqrt{p(x)}} \left[F \exp\left(\dfrac{\mathrm{i}}{\hbar}\displaystyle\int_{x_2}^{x} \mathrm{d}x'\, p(x')\right)\right], & x > x_2. \end{cases}$$

Do *not* assume $C = 0$. Calculate the tunneling probability $T = |F|^2/|A^2|$, and show that your result reduces to Eq. (7) in the case of a broad, high barrier.