Problem Set
Quantum statistics

1 Two-dimensional electron gas

A confined electron gas can form at the interface between two doped semiconductors (e.g., GaAs/AlGaAs). The confinement is such that one can consider that the gas is strictly two-dimensional. Electron-electron interactions will be neglected in the following and we will adopt the effective mass approximation. We call n the electronic density of the gas and $A = L_x L_y$ its surface (which we assume to be very large as compared to all the other length scales of the problem). Here, L_x and L_y are the lateral dimensions of the gas in the x and y directions, respectively. We recall that the electrons are spin-1/2 fermions, and thus obey the Fermi–Dirac statistics. The average occupancy of an energy state ϵ is then given by the Fermi–Dirac distribution function

$$f(\epsilon) = \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} + 1}, \tag{1}$$

where $\beta = 1/k_\mathrm{B}T$, with T the temperature of the gas, and where $\mu = \mu(T)$ is the chemical potential.

(a) Plot the Fermi–Dirac distribution (1). In particular, analyze the $T = 0$ case.

(b) Using periodic boundary conditions (why can you do so?), solve Schrödinger's equation and show that the electronic dispersion is given by

$$\epsilon_\mathbf{k} = \frac{\hbar^2 |\mathbf{k}|^2}{2m},$$

where the wavevector $\mathbf{k} = (k_x, k_y)$ is quantized according to $k_x = 2\pi n_x/L_x$ and $k_y = 2\pi n_y/L_y$, with n_x and n_y integer numbers.

(c) Show that the electronic density of states $\rho(\epsilon)$ is energy-independent and is given by $\rho(\epsilon) = 1/\Delta$, where $\Delta = \pi\hbar^2/mA$.

(d) Give an expression for the average number N of electrons in the gas. Deduce from the previous result that the chemical potential reads

$$\mu(T) = k_\mathrm{B}T \ln\left(\mathrm{e}^{T_\mathrm{F}/T} - 1\right),$$

where T_F is the Fermi temperature, defined through the Fermi energy as $E_\mathrm{F} = k_\mathrm{B}T_\mathrm{F}$. What is the definition of the Fermi energy? Give an expression of E_F as a function of N and Δ. Interprete this result. Plot μ as a function of T.

(e) Give a formal expression of the average energy E of the system in terms of an integral over ϵ, that we will not explicitly calculate. Show that the grand-canonical potential reads

$$\Omega = -\frac{k_\mathrm{B}T}{\Delta} \int_0^\infty \mathrm{d}\epsilon\, \ln\left(1 + \mathrm{e}^{-\beta(\epsilon-\mu)}\right).$$

Deduce from the previous two results that $\Omega = -E$.

(f) Show that the two-dimensional pressure P of the gas is related to the average energy via the expression $P = E/A$.

(g) Using your answers to questions (e) and (f) above, derive the equation of state at $T = 0$.

(h) At low temperature ($T \ll T_{\mathrm{F}}$), expand the average energy to second order in T/T_{F} so as to obtain the equation of state. Notice that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{x^2 \mathrm{e}^x}{(\mathrm{e}^x+1)^2} = \frac{\pi^2}{3}.$$

(i) Calculate the equation of state at high temperature ($T \gg T_{\mathrm{F}}$). Comment your result.

(j) *(Optional question)* Calculate now the equation of state for an arbitrary temperature. One gives

$$\int_0^{\infty} \mathrm{d}x \frac{x}{\mathrm{e}^x/a+1} = -\mathrm{Li}_2(-a),$$

where a is a constant, and where $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ is the polylogarithm function of order s.

(k) Compare all the results of this problem to the three-dimensional case encountered in the lecture.

2 Bose–Einstein condensation

Let us consider a system of N bosons with mass m and spin s ($s \in \mathbb{N}$) occupying a volume V. In the following, the interactions between the bosons are neglected. Accessible energy levels are denoted by $\epsilon_{\mathbf{k}}$, and the ground state energy is set to zero.

(a) What is the average occupancy $n(\epsilon)$ of a state of energy ϵ at temperature T? Show that the density of states takes the form $d(\epsilon) = KV\sqrt{\epsilon}$, where K is a constant. Give the expression for K. What is the sign of the chemical potential μ? How is μ determined in the thermodynamic limit?

(b) Plot on the same graph $n(\epsilon)$ as a function of ϵ for two different chemical potentials $\mu_1 < \mu_2$ while T is being kept fixed. On another graph, plot $n(\epsilon)$ as a function of ϵ for two different temperatures $T_1 < T_2$ while μ is being kept fixed. Considering that the number of particles is fixed, show that

$$\left(\frac{\partial \mu}{\partial T}\right)_N < 0.$$

(c) By introducing the fugacity $\varphi = \mathrm{e}^{\beta\mu}$ as well as the function

$$f(\varphi) = \int_0^{\infty} \mathrm{d}x \frac{\sqrt{x}}{\mathrm{e}^x/\varphi - 1},$$

determine graphically the chemical potential μ. What happens when the temperature is lowered? Show that there exists a critical temperature T_{B}, called the *Bose temperature*, for which $\mu = 0$. Note that

$$\int_0^{\infty} \mathrm{d}x \frac{\sqrt{x}}{\mathrm{e}^x - 1} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right),$$

where $\zeta(z)$ is the Riemann zeta function, which is defined for any complex number z such that $\mathrm{Re}(z) > 1$ by the Riemann series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. In particular, $\zeta(3/2) \simeq 2.61$ and $\zeta(5/2) \simeq 1.34$.

(d) We now consider that $T < T_{\mathrm{B}}$ and we assume N to be fixed. Show that the number of particles in the ground state is given by

$$N_0 = N\left[1 - \left(\frac{T}{T_{\mathrm{B}}}\right)^{3/2}\right].$$

Is it possible to condensate photons?

(e) Give an expression of the average energy E of the system in terms of an integral over ϵ that we will not explicitly calculate. Show that the grand-canonical potential Ω reads

$$\Omega = KVk_{\mathrm{B}}T \int_0^{\infty} \mathrm{d}\epsilon \sqrt{\epsilon} \ln\left(1 - \mathrm{e}^{-\beta(\epsilon-\mu)}\right).$$

Deduce from the two previous results that $\Omega = -2E/3$. Then, show that the pressure of the Bose gas is given by $P = 2E/3V$.

(f) Derive an expression for the pressure of the system at $T < T_{\mathrm{B}}$. Note that

$$\int_0^{\infty} \mathrm{d}x \frac{x^{3/2}}{\mathrm{e}^x - 1} = \frac{3\sqrt{\pi}}{4}\zeta\left(\frac{5}{2}\right).$$

(g) We now consider that T is kept constant, instead of V (N remains fixed throughout). What happens when the volume of the system is decreased? Show that the Bose condensation takes place for

$$V_{\mathrm{B}} = \frac{1}{(2s+1)\zeta(3/2)} N\Lambda_T^3,$$

where $\Lambda_T = (2\pi\hbar^2/mk_{\mathrm{B}}T)^{1/2}$ is the thermal de Broglie wavelength. Plot a few isothermal curves in a P-V diagram. Discuss your results.

(h) Liquid ^{4}He presents a superfluid transition at 2.17 K. Compare such an experimental result to the Bose temperature. Parameters for liquid ^{4}He are: spin $s = 0$, density $0.12\,\mathrm{g/cm^3}$, and $m = 4\times m_{\mathrm{proton}} = 6.7\times10^{-27}$ kg. We recall that $\hbar = 1.0\times10^{-34}$ J.s and $k_{\mathrm{B}} = 1.4\times10^{-23}$ J/K.