Exercises on the general relativity course

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Abstract

I. EX1

We perform a change of basis of a vector space E_2, defined by:

$$\vec{e}_1' = 3\vec{e}_1 + \vec{e}_2 \ \ ; \ \ \vec{e}_2' = -\vec{e}_1 + 2\vec{e}_2 \ . \tag{1}$$

- Starting from the definition of the covariant components of the fundamental tensor g_{ij}, calculate its new components g'_{ij} as a function of the old ones.
- Verify the results using the general formula of a change of basis for a second rand tensor.

II. EX2

The mixed components t^i_{jk} of a tensor $\mathbb{T}$, belonging to the tensor product space $E_2^{(3)}$ are as follows:

$$t^1_{11} = 0 \ , \ t^1_{12} = 2 \ , \ t^1_{21} = -1 \ , \ t^1_{22} = 3 \ , \ t^2_{11} = 1 \ , \ t^2_{12} = -1 \ , \ t^2_{21} = 0 \ , \ t^2_{22} = -2 \ . \tag{2}$$

- Calculate the contracted components $u_k = t^i_{ik}$ of the tensor $\mathbb{T}$. Write the expression for the tensor $\mathbb{U}$ of components u_k.
- We give ourselves a basis $\{\vec{e}_i\}$ of E_2 in which the fundamental tensor g_{ij} has as matrix:

$$[g_{ij}] = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 1 \end{pmatrix} \tag{3}$$

  Determine the covariant components t_{ijk} of the tensor $\mathbb{T}$.
- Determine the contravariant components g^{ij} of the fundamental tensor.
- Calculate the mixed components t^{ij}_k of the $\mathbb{T}$ tensor.

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III. EX3

A point M is represented in cylindrical coordinates by the variables ρ, φ, z.

- Determine the expression of the position vector $\vec{OM}(\rho, \varphi, z)$ of any point M on the Cartesian basis $\{\vec{i}, \vec{j}, \vec{k}\}$.
- Determine the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ of the natural basis and represent them on a graph.
- Show that these vectors are orthogonal to each other.
- Calculate the norms of the vectors in the natural basis.
- Determine the linear element of E_3.
- Determine the volume element (jacobian of the transformation).

IV. EX4

Prove the transformation formula for Christoffel symbols:

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m} \, . \tag{4}$$

V. EX5

The spherical coordinates are defined by: $(x = r \sin\theta \cos\varphi, y = \sin\theta \sin\varphi, z = r\cos\theta)$.

- Calculate the line element ds^2.
- Obtain the components of the metric tensor g_{ij}.
- Calculate the Christoffel symbols of the first kind in spherical coordinates.
- Calculate those of the second kind.

VI. EX6

A particle moves along a trajectory defined in spherical coordinates (r, θ, φ). Determine the contravariant components a^k of the acceleration $\vec{a}$ of this particle for the following trajectories:

- The trajectory is defined by: $r = c, \theta = \omega t, \varphi = \pi/4$ where t is the time.
- The trajectory is defined by: $r = c, \theta = \pi/4, \varphi = \omega t$. Calculate the norm of the acceleration and show that we find the back classic formula: $\|\vec{a}\| = r\omega^2$.